Exam: Introduction to Risk Theory (191515101) 2023-2024 M2

Mon 29-01-2024 (08.45-11.45) **Instructor: Ashoke Sinha**

Instructions:

- The exam consists of 4 exercises containing total 35 points.
- Do not write on the formula sheet and the table. Please return them at the end of the exam.
- All answers/calculations must be motivated.

Exercise 1. A customer of an insurance company has a total wealth of 100 (in some units) and is facing a random loss X that is uniformly distributed on $[0, 100]$. The utility function of the customer is $u(x) = \sqrt{x}$.

(a) Show that the maximal premium the person would be willing to pay to insure him/herself against the loss X is 55.56 (unit of money). [3]

(b) If an insurance company fixes its premium using the standard deviation principle with a loading factor $\alpha := \frac{\sqrt{3}}{2}$, will there be a business between the insurer and the person? [2]

(c) Recall that a premium principle $\pi[\cdot]$ is said to be *consistent* if

$$\pi[X + c] = \pi[X] + c, \quad \text{for every constant } c.$$

Is the premium principle used by the insurer consistent ? [2]

Exercise 2. A certain risk-portfolio contains two types of contracts. For type k, $(k = 1, 2)$, the probability that a claim will appear is q_k and the number of policies is n_k. All policies are considered to be independent. Also, when there is a claim of type k, the probability is $p_k(x)$ that the claim-amount is x. These quantities are summarized in the following table:

k	n_k	q_k	$p_k(1)$	$p_k(2)$	$p_k(3)$
Type 1	1000	0.01	0.5	0	0.5
Type 2	2000	0.01	0.5	0.5	0

(a) Give an exact-model of the total risk S for the given risk-portfolio, with properly defined random variables. [2]

(b) Is your model an individual one or a collective one? Explain. Also, describe briefly the difference between the two? [2]

(c) Show that the expectation and variance of the total risk are 50 and 99.15, respectively. [3]

(d) Use the normal approximation (CLT) to determine the minimal safety loading (relative loading factor) θ an insurance company should use to make sure that the chance of a loss is at most 0.5%. [3]

(e) One (other) way to model/interpret the given risk-portfolio is as follows. There are total 3000 clients, each having a probability of 0.01 of submitting a (positive) claim. So, altogether there will be N number of claims, where $N \sim \text{Binomial}(3000, 0.01)$. One such individual claim W can be either 1, 2 or 3 with probabilities proportional to how often they may appear, leading to $P(W = 1) = \frac{3}{6} = \frac{1}{2}$, $P(W = 2) = \frac{2}{6} = \frac{1}{3}$ and $P(W = 3) = \frac{1}{6}$.

Using this interpretation, denote the total risk from the portfolio by T and give a suitable mathematical model for T. What is the name of this model? [2]

Exercise 3. Recall that in a *discrete time* risk process, for example, when an insurance company evaluates its financial situation at each year-end, the wealth of the company at year $n \geq 1$ is given by $U(n) = u + G_1 + G_2 + \cdots + G_n$, where u is the initial capital and G_i is the gain (profit) in year i, with G_i's assumed to be i.i.d. with distribution F_G, say.

In this case, the ruin probability is related to the discrete risk adjustment coefficient $\tilde{R}$, which is the non-zero solution (for r) to the equation

$$m_G(-r) = 1,$$

where m_G is the moment generating function (*mgf*) corresponding to F_G.

Now consider an insurance company that faces, in year $i = 1, 2, \ldots$, a total claim of W_i. Assume that W_i's are i.i.d. $N(5, 1)$-distributed. The company uses a loading factor of $\theta = 0.25$ in deciding its premium. After collecting the premium, the insurer decides to buy a reinsurance, where the reinsurer covers a fraction α $(0 < \alpha < 1)$ of the total claims. To fix its premium the reinsurer uses the loading factor of $\xi = 0.4$.

In the following, you will analyze this insurance portfolio (modelled in discrete time).

(a) Suppose $X \sim N(\mu, \sigma^2)$, and c and a are two real numbers with $a \neq 0$. Use the *mgf*-technique to find the probability distribution of the3 random variable $Y := c + a\,X$. [2]

(b) Show that the risk adjustment coefficient for the reinsured portfolio is given by [5]

$$\tilde{R} = \frac{5 - 8\alpha}{2(1-\alpha)^2}.$$

(c) What is the best choice of α as far as the ruin probability is concerned? [3]

Exercise 4. An automobile insurance company has a bonus-malus system with four levels of premium level 1 corresponds to the highest premium and level 4 the lowest. The clients in these levels pay, respectively, 120%, 100%, 80% and 50% of a certain base premium. The rules for moving between these levels are:

- If no claim is made in a year, the policyholder moves to the next higher level (i.e. with less premium), if possible; otherwise, remains at the highest level.
- If one claim is made in a year, the policyholder moves to the next lower level (i.e., with more premium), if possible; otherwise, remains at, the lowest level.
- If more than one claim is made in a year, then the policyholder moves to the lowest level.

The yearly number of claims N by a policyholder is distributed geometrically with probability mass function $P(N = n) = p^n(1-p)$, $n = 0, 1, 2, \ldots$.

(a) Construct an appropriate Markov chain with the levels being the states and derive its transition matrix $\mathbf{P}$. [4]

<u>You do not need to</u>, but it can be shown that the steady state distribution for this chain is of the form: $\pi.= [kp^2(1+p-p^2),\ kp^2(1-p)(2-p),\ kp(1-p)^2,\ k(1-p)^3]$, for some constant k.
In particular, for a policyholder with $p = 0.1$, $\pi = [0.0109\,k,\ 0.0171\,k,\ 0.081\,k,\ 0.729\,k]$.

(b) Suppose the base premium used by the insurance company is 1000 euro and a particular policyholder has $p = 0.1$. Find the premium the policyholder is expected to pay in the0 steady-state. Motivate your answer (and the answer should be a number). [2]