# Analysis 2

## Chapter 4: Differentiability of functions in one variable

Section ?

4.1 => **Definition**: Let $I$ be an interval with more than one point and $f: I \to \mathbb{R}$. Then $f$ is called differentiable at a point $x \in I$ if $\lim_{t \to x} \frac{f(t)-f(x)}{t-x}$ exists as real number and we call this limit the derivative of $f$ at $x$, denoted by $f'(x)$, $\frac{d}{dx} f(x)$.

4.2 => Facts:

1) $\frac{f(t)-f(x)}{t-x}$ is called the differential quotient between point $t$ and $x$ $(t \neq x)$
2) $f'(x)$ exists if and only if $\lim_{t \to x^-} \frac{f(t)-f(x)}{t-x} = f'^-(x)$ and $\lim_{t \to x^+} \frac{f(t)-f(x)}{t-x} = f'^+(x)$ exists and are equal (consequence of facts on limits.
3) $f$ is differentiable $\Leftrightarrow \forall (t_n)_{n \in \mathbb{N}^*} \in I \setminus \{x\}$ converging to $x$: $\left(\frac{f(t_n)-f(x)}{(t_n)-x}\right)$ converges to a limit independent of the sequence $(t_n)_{n \in \mathbb{N}^*}$
4) Differentiability of $f$ is a local propertie, that means $f$ differentiable at $x \Leftrightarrow f|(-\delta+x, \delta+x)$ is differentiable at $x$.

4.3 => Examples:

a) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$, $n \in \mathbb{N}^*$

Let $x \in \mathbb{R}$ $\frac{f(t)-f(x)}{t-x} = \frac{t^n - x^n}{t-x} = \frac{(t-x)(t^{n-1}+t^{n-2}x+\ldots+tx^{n-2}+x^{n-1})}{t-x}$

$= t^{n-1} + t^{n-2}x + \ldots + tx^{n-2} + x^{n-1}$

$\Rightarrow \lim_{t \to x} \underbrace{t^{n-1}}_{x^{n-1}} + \underbrace{t^{n-2}x}_{x^{n-1}} + \ldots + \underbrace{tx^{n-2}}_{x^{n-1}} + \underbrace{x^{n-1}}_{x^{n-1}} = nx^{n-1} = f'(x)$ by rules for continuity limits

b) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = c$, $c \in \mathbb{R}$

$\Rightarrow f$ is differentiable at every $x \in \mathbb{R}$ with $f'(x) = 0$

c) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$, $x \in \mathbb{R}$

$f$ is not differentiable at $x=0$ and differentiable at $x \neq 0$

$f'(x) = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$ by fact 4.2 (4) and (a)

Calculate $f'^+(0)$ and $f'^-(0)$

$\lim_{t \to 0^+} \frac{f(t)-f(x)}{t-x} = \frac{f(t)-f(0)}{t-0} = \frac{t}{t} = 1$, $\lim_{t \to 0^-} \frac{f(t)-f(0)}{t-0} = \frac{-t}{t} = -1$

$\Rightarrow f$ not differentiable at $x=0$

d) $\exp: \mathbb{R} \to \mathbb{R}$, $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\forall x \in \mathbb{R}$ $\lim_{h \to 0} \frac{\exp(x+h) - \exp(x)}{h} = \exp(x)$

$\overset{(x+h:=t)}{=} \lim_{t \to x} \frac{\exp(t) - \exp(x)}{t-x} = \exp'(x)$

This follows from uniform convergence on compact intervals $\sum_{n=0}^{\infty} f_n$, $f_n(x) = \frac{x^n}{n!}$, $n \in \mathbb{N}^* \cup \{0\}$

e) $\sin'(x) = \cos(x)$, $\cos'(x) = -\sin(x)$ $\forall x \in \mathbb{R}$ (ps7 Analysis 1)

4.4 => proposistion: Differentiable => Continuous

Let $I$ be an interval with more than one point and $f: I \to \mathbb{R}$ be differentiable at $x \in I$. Then $f$ is continuous at $x$.

↳ proof: $\lim_{t \to x} (f(t) - f(x)) = \lim_{t \to x} \left[ \frac{f(t)-f(x)}{t-x} \cdot (t-x) \right] \underset{f \text{ diff.}, \lim_{t \to x} t-x = 0}{=} \lim_{t \to x} \frac{f(t)-f(x)}{t-x} \cdot \lim_{t \to x} t-x$

$= f'(x) \cdot 0 = 0$

4.5 => Remark: A countinuous function need not be differentiable (see example 4.3.(c))

4.6 => proposistion: Sumrule, product rule & quotient rule

Let $f, g: I \to \mathbb{R}$, $I$ interval with more than one point and $f, g$ differentiable at $x \in I$. Then:

1) $f+g$ is differentiable at $x$ and $(f+g)'(x) = f'(x) + g'(x)$

↳ proof: $\lim_{t \to 0} \left[ \frac{f(x+t) - f(x)}{t} \right]$, $\lim_{t \to 0} \left[ \frac{f(x+t) + g(x+t) - (f(x) + g(x))}{t} \right]$

$= \lim_{t \to 0} \frac{f(x+t) + g(x+t) - f(x) - g(x)}{t} = \lim_{t \to 0} \frac{f(x+t) - f(x) + g(x+t) - g(x)}{t}$

$= \lim_{t \to 0} \frac{f(x+t) - f(x)}{t} + \lim_{t \to 0} \frac{g(x+t) - g(x)}{t} = f'(x) + g'(x)$

2) $f \cdot g$ is differentiable at $x$ and $(f \cdot g)'(x) = f'(x) g(x) + f(x) g'(x)$

↳ proof: $\lim_{t \to x} \frac{(f \cdot g)(t) - (f \cdot g)(x)}{t-x} = \lim_{t \to x} f(t) \frac{g(t) - g(x)}{t-x} + \lim_{t \to x} \frac{f(t) - f(x)}{t-x} g(x)$

$= \lim_{t \to x} f(t) \cdot \lim_{t \to x} \frac{g(t) - g(x)}{t-x} + g(x) \lim_{t \to x} \frac{f(t) - f(x)}{t-x}$

$= f(x) g'(x) + f'(x) + g(x)$

3) If $g(x) \neq 0$ then $\frac{f}{g}$ is differentiable and $\left(\frac{f}{g}\right)'(x) = \frac{f'(x) g(x) - f(x) g'(x)}{g(x)^2}$

↳ proof: $\frac{\frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}}{t-x} = \frac{1}{g(t) g(x)} \left[ \frac{g(x)(f(t) - f(x))}{t-x} - \frac{g(t) - g(x)}{t-x} f(x) \right]$

$t \to x$ by continuity of $f$ and $g$ and the assumption we get the claim

4.7 $\Rightarrow$ Example:

f) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$, $n \in \mathbb{Z}$. $f|_{(-\infty,0)\cup(0,\infty)}$ is differentiable at $x \in \mathbb{R}\setminus\{0\}$

$n<0$ $f(x) = \frac{1}{x^{-n}} = \frac{h(x)}{g(x)}$ with g and h differentiable at x

$\overset{\text{quotient rule}}{\Longrightarrow}$ $f'(x) = \frac{-(-1)n x^{-n-1}}{x^{-n} \cdot x^{-n}} = n \cdot x^{n-1}$

g) Any rational function $r(x) = \frac{h(x)}{g(x)}$, h and g polynomials is differentiable at $x \in \mathbb{R}$ if $p(x) \neq 0$

4.8 $\Rightarrow$ proposistion: Chain rule

Let I and J be intervals with more than one point. $f: I \to \mathbb{R}$ differentiable at x and $h: J \to \mathbb{R}$ differentiable at $f(x)$ with $J \supseteq \{f(y) : y \in I\}$

Then $h \circ f$ is diffentiable at x with $(h\circ f)'(x) = h'(f(x)) \cdot f'(x)$

↳ proof: define $\varphi(s) = \left\{\begin{matrix} \frac{h(s)-h(f(x))}{s-x} & \text{if } s \neq x \\ h(f'(x)) & \text{if } s = x \end{matrix}\right\}$ $\varphi: J \to \mathbb{R}$

$\lim_{s\to x} \frac{h(f(t)) - h(f(x))}{s-x} = \lim_{s\to x} \frac{h(f(s)) - h(f(x))}{f(s) - f(x)} \cdot \frac{f(s)-f(x)}{s-x} = h'(f(x)) \cdot f'(x)$

4.9 $\Rightarrow$ Example:

h) let $f: I \to J$, where I and J are intervals with more than one element, bijective, continuous and such that $f^{-1}$ continuous

if f is differentiable at some $x \in I$ and $f'(x) \neq 0$, then $f^{-1}$ is differentiable at $f(x)$ and $(f^{-1})'(f(x)) = \frac{1}{f'(x)}$

↳ Fake proof: using chainrule $(h\circ f)'(x) = h'(f(x)) \cdot f'(x)$

apply this to $h = f^{-1}$ and $f = f$ $\Rightarrow (h\circ f)(x) = f^{-1}(f(x)) = x$

$(h\circ f)'(x) = 1$

$h'(f(x) \cdot f'(x)) = (f^{-1})(f(x)) \cdot f'(x) \Rightarrow (f^{-1})'(f(x)) = \frac{1}{f'(x)}$

↳ True proof: let $(t_n)_{n\in\mathbb{N}^*}$ be a sequence in $J\setminus\{f(x)\}$, $t_n \xrightarrow{n\to\infty} f(x)$

$\overset{h = f^{-1} \text{ cont.}}{\Longrightarrow}$ $h(t_n) \xrightarrow{n\to\infty} h(f(x)) = x$

$\frac{f^{-1}(t_n) - f^{-1}(f(x))}{t_n - f(x)} = \frac{h(t_n) - x}{f(h(t_n)) - f(x)} = \left(\frac{f(h(t_n)) - f(x)}{h(t_n) - x}\right)^{-1}$

$\left\{\begin{matrix} h(t_n) = f^{-1}(t_n) \\ \Rightarrow f(h(t_n)) = t_n \end{matrix}\right\}$

Since f is differentiable $\overset{4.2(2)}{\underset{h\text{ cont.}}{\Longrightarrow}}$ $\lim_{n\to\infty} \frac{f(h(t_n)) - f(x)}{h(t_n) - x}$ exists and equals $f'(x)$

$\overset{f'(x)\neq 0}{\Longrightarrow}$ $\lim_{n\to\infty} \frac{f^{-1}(t_n) - f^{-1}(f(x))}{t_n - f(x)} = \frac{1}{f'(x)}$

i) $f: [0,\infty) \to \mathbb{R}$, $x \mapsto \sqrt{x}$ see tutorial

$\ln: (0,\infty) \to \mathbb{R}$, (inverse of exp)

4.10 => **Definition**: For $f: I \to \mathbb{R}$ (or $\mathbb{C}$) differentiable (on $I$) the function $f': I \to \mathbb{R}$ (or $\mathbb{C}$) $: x \mapsto f'(x)$ is called the (first) derivative of $f$
Recursively, define for $n \in \mathbb{N}^*$ $f^{(n+1)} := (f^{(n)})' := \frac{d^{n+1}}{dx^{n+1}} f$, where $f^{(n)}$ is differentiable and $f^{(1)} = f'$ $(f'' = f^{(2)})$
We also say that $f$ is $(n+1)$-times differentiable (in that case)

=> Note: In order to define $f^{(n+1)}(x)$ for some $x \in I$, we need that $f^{(n)}(y)$ exists for a subinterval $I$ including $x$ (with more than one point) $\frac{f^{(n)}(y) - f^{(n)}(x)}{y-x}$

4.11 => **Definition**: let $f: I \to \mathbb{R}$ (or $\mathbb{C}$) be differentiable. We say that $f$ is continuously differentiable iff $f'$ is continuous.
The set of continuously differentiable functions (on $I$) is denoted by $C^1(I;\mathbb{R}) = C^1(I)$
↳ for n'th times: $C^n(I) = C^n(I;\mathbb{R}) = \{f: I \to \mathbb{R}/\mathbb{C} \mid f$ n-times differentiable and $f^{(n)}$ continuous$\}$
"smooth functions" $C^\infty(I) := C^\infty(I;\mathbb{R}) = \bigcap_{n \in \mathbb{N}^*} C^n(I;\mathbb{R}/\mathbb{C})$

4.12 => Example:
i) $\exists f: I \to \mathbb{R}$ differential, but $f'$ not continuous
$f: [0,\infty) \to \mathbb{R}$, $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable
$\left.\begin{array}{l} f'(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x}), x > 0 \\ f'(x) = 0 \end{array}\right\}$ $f'$ not continuous

## Section 2: Mean value theorem

4.13 => **Definition**: let $(X,d)$ metric space, $X \neq \emptyset$ and let $x_0 \in X$
$f: X \to \mathbb{R}$, then we say that $f$ has a local maximum/minimum if $\exists r > 0$, $\forall y \in \{x \in X : d(x,x_0) < r\}$ : $f(y) \leq f(x_0)$ / $f(y) \geq f(x_0)$ and $f(x_0)$ is called a local maximum/minimum or extremum

4.14 => proposition: Let $f: I \to \mathbb{R}$ differentiable and assume that $x_0 \in I$ is not a boundry point of $I$
if $f$ has a local extremum at $x_0$, then $f'(x_0) = 0$

↳ proof: Suppose that $f$ has local minimum at $x_0$ ($\Rightarrow f(t) \geq f(x_0)$)

$t < x_0 \Rightarrow t - x_0 < 0 \Rightarrow \frac{f(t)-f(x_0)}{t-x_0} \leq 0$

$t > x_0 \Rightarrow t - x_0 > 0 \Rightarrow \frac{f(t)-f(x_0)}{t-x_0} \geq 0$

$\Rightarrow \lim_{t \to x_0^-} \frac{f(t)-f(x_0)}{t-x_0} = f'^-(x_0) \leq \wedge \lim_{t \to x_0^+} \frac{f(t)-f(x_0)}{t-x_0} = f'^+(x_0) \geq 0$

$\overset{f \text{ diff.}}{\Rightarrow} f'(x_0) = 0$

4.15 ⇒ Corrolary: let $a<b$ real numbers $f: [a,b] \to \mathbb{R}$ differentiable on $(a,b)$ (this means $f|(a,b): (a,b) \to \mathbb{R}$ is differentiable) and continuous on $[a,b]$ with $f(a) = f(b)$ Then $\exists x_0 \in [a,b]: f'(x) = 0$

↳ proof: Since $f$ is continuous $\exists x_+, x_- \in [a,b]: \forall t \in [a,b]: f(x_-) \leq f(t) \leq f(x_+)$

If $x_-$ and $x_+$ are in $\{a,b\}$, then by $f(a) = f(b)$ we get $f$ is constant

If not then $x_-$ or $x_+ \in (a,b)$, apply prop 3.14 $\Rightarrow f'(x_+)$ or $f'(x_-) = 0$

4.16 ⇒ Theorem: Mean value theorem

Let $f: [a,b] \to \mathbb{R}$, $a<b$ real, $f|(a,b)$ differentiable, $f$ continuous then $\exists \xi \in (a,b): \frac{f(b)-f(a)}{b-a} = f'(\xi)$

↳ proof: see tutorial

Consequence of theorem: if $f' = 0$ on $I \Rightarrow f$ constant

4.17 ⇒ Theorem: Mean value theorem (MVT) - general

Let $a<b$ real numbers and $f, g: [a,b] \to \mathbb{R}$ continuous, $g|(a,b)$, $f|(a,b)$ differentiable and $g(x) \neq 0$ $\forall x \in [a,b]$. Then $\exists \xi \in (a,b): \frac{f(b)-f(a)}{b-a} = \frac{f'(\xi)}{g'(\xi)}$

↳ proof: define $F: [a,b] \to \mathbb{R}: F(b) = F(a) = 0$

$\Rightarrow \exists \xi \in (a,b): F'(\xi) = 0$ (see 4.15)

$F(t) = f(t) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)} \cdot (g(t) \cdot g(a))$

$F: [a,b] \to \mathbb{R} \Rightarrow F(b) - F(a) = 0$

$F$ is continuous $F|(a,b)$ differentiable $\overset{4.15}{\Rightarrow} \exists \xi \in (a,b): F'(\xi) = 0$

$F'(t) = f'(t) - \frac{f(b)-f(a)}{g(b)-g(a)} \cdot g'(t)$ $(g(b) - g(a) \neq 0)$

with more than one point

4.18 ⇒ Corrolary: let $I = (a,b)$ be an interval (w.m.t.o.p), $f[a,b] \to \mathbb{R}$ continuous, $f: (a,b) \to \mathbb{R}$ differentiable. Then:

1) $f' = 0$, $f'(x) = c$ $\forall x \in I \Rightarrow f$ is constant on $I$

2) $f' \overset{>}{\geq} 0$, $f'(x) \overset{>}{\geq}$ $\forall x \in I \Rightarrow f$ is increasing/non-decreasing on $I$

3) $f' \overset{<}{\leq} 0$, $f'(x) \overset{<}{\leq}$ $\forall x \in I \Rightarrow f$ is decreasing/non-increasing on $I$

↳ proof 2: let $x, y$ be in $(a,b)$, $x<y$ and $f' > 0$

$\overset{MVT}{\Rightarrow} \frac{f(y)-f(x)}{y-x} = \underbrace{f'(\xi)}_{>0}$ for some $\xi \in (x,y)$

$\Rightarrow f(y) - f(x) > 0 \Rightarrow f(y) > f(x) \Rightarrow f$ is increasing

1) follows from 2&3

4.19 $\Rightarrow$ Example: let $f: \mathbb{R} \to \mathbb{R}$, $x \to x^3$. It's easy to see that $f$ is increasing but $f'(x) = 3x^2 \Rightarrow f'(0) = 0$

$\Rightarrow f'(x) \not> 0 \ \forall x \in \mathbb{R}$

4.20 $\Rightarrow$ Remark: For $f: I \to \mathbb{C}$ ($I$ and $f$ as in 4.18) and $f' = 0$ on $I$

$\Rightarrow f$ constant on $I$ (by considering $(\mathrm{Re} f)(x) = \mathrm{Re}(f(x))$, $(\mathrm{Im} f)(x) = \mathrm{Im}(f(x))$, which are satisfying the same assumptions as $f$.

(plays a role for exponential law, see ps 1)

4.21 $\Rightarrow$ proposistion: let $f: I \to \mathbb{R}$ be differentiable, $I$ interval (w.m.1.ap)

then $f'(I)$ is an interval

($\tilde{I} \subseteq \mathbb{R}$ is an interval $\Leftrightarrow \forall x, y \in \tilde{I}, x < y: [x, y] \subset \tilde{I}$)

↳ proof: show that for $x, y \in I$ with $f'(x) < f'(y)$:

$\forall c \in (f'(x), f'(y)): \exists \xi \in I: f'(\xi) = c$ and MVT

4.22 $\Rightarrow$ Theorem: de L'Hospital

let $g, f: (a, b) \to \mathbb{R}/\mathbb{C}$ where $-\infty \leq a < b \leq +\infty$ differentiable and $g'(x) \neq 0$ for all $x$ suff. close to $a$ ($\exists \delta > 0\ g'(x) \neq 0\ \forall x \in (a, a+\delta)$)

and suppose that either $\lim_{t \to a^+} f(t) = \lim_{t \to a^+} g(t) = 0$ or

$\lim_{t \to a^+} g(t) = +\infty$

and suppose that $\lim_{t \to a^+} \frac{f'(t)}{g'(t)} = A$ exists in $\mathbb{R}$ or $+\infty$

Then $\lim_{t \to a^+} \frac{f(t)}{g(t)}$ exists and equals $A$.

Analogausly the statement holds if $t \to a^+$ gets replaced by $t \to b^-$ and $g'(x) \neq 0$ suff. close to $b$. $\lim g(t) = \infty$

↳ proof: $g(x) \neq 0$ "close to $a$": we can assume w.l.o.g that $g'(x) \neq 0$ $\forall x \in (a, b) \Rightarrow \exists b' \in (a, b)$: $g$ has no zero in $(a, b')$. Set $b = b'$

- If "$\frac{?}{\infty}$" case; then $g$ strictly decreasing
- If $A < +\infty$, let $\alpha \in (A, \infty)$ and $r \in (A, \alpha)$

$\Rightarrow \exists c > a: \frac{f'(t)}{g'(t)} < r$ for $t \in (a, c)$ $\quad \frac{f(y)}{g(y)} \leq r < \alpha$ *

let $x, y \in (a, c): x < y: \frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} < r$ for some $t \in (x, y)$ MVT

Assume "$\frac{0}{0}$" case and let $x \to a^+$

Do the same for $-f, -A$ instead of $f, A$ (mult. with $-1$)

$\Rightarrow \forall B \in (-\infty, A), \tilde{r} \in (B, A)\ \exists \tilde{c} > a: \frac{f(y)}{g(y)} \geq \tilde{r} > B \quad \forall y \in (a, \tilde{c})$ **

$\overset{*,**}{\Rightarrow} \lim_{y \to a^+} \frac{f(y)}{g(y)} = A$

## Section 3: Elementary facts (exp, ln, sin, cos)

=> Recal: $\exp: \mathbb{R} \to \mathbb{R}$ $\quad x \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ is well-defined
$\exp_{\mathbb{C}}: \mathbb{C} \to \mathbb{C}$ (series con. abs. $\forall x$, uniformly on compact sets)

4.23 => proposistion:
- $\exp(0) = 1$, $\exp(x) > 0$ $\forall x \geq 0$, $x \in \mathbb{R}$
- $\forall x \geq 0$, $x \in \mathbb{R}$ $\exp(x) \geq 1+x$
- exp differentiable (=> continuous) on $\mathbb{R}$, $\exp' = \exp$
  => $\exp \in C^\infty(\mathbb{R}, \mathbb{R})$ (tutorial 0)
- $\psi: \mathbb{R} \to \mathbb{C}$ $t \mapsto \exp(at)$ $a \in \mathbb{C}$ is differentiable, $C^\infty(\mathbb{R};\mathbb{C})$, $\psi'(t) = a\exp(at)$
- $\forall x, y \in \mathbb{C}: \exp(x+y) = \exp(x)\exp(y)$ (tutorial 1) "exponential law"
- $\forall x \in \mathbb{C}$: => $\exp(x) \neq 0 \wedge \exp(-x) = \frac{1}{\exp(x)}$
  => $\exp(\mathbb{R}) \subseteq (0,\infty)$ Moreover $\lim_{x \mapsto \infty} \exp(x) = +\infty$, $\lim_{x \to -\infty} \exp(x) = 0$
- $\exp(\mathbb{R}) = (0,\infty)$ (by intermediate value theorem)
- exp is increasing (strictly increasing) by $\exp' = \exp > 0$ on $\mathbb{R}$
- $\exp: \mathbb{R} \to (0,\infty)$ is bijective (by $\exp(\mathbb{R}) = (0,\infty) \wedge \exp$ increasing)
- exp is invertible and the inverse is continuous
  ($f: [a,b] \to f[a,b]$ bijective, continuous => $f^{-1}$ continuous)
  We call that inverse the natural logaritm $\ln: (0,\infty) \to \mathbb{R}$
  => $\ln \in C^\infty(0,\infty;\mathbb{R})$ (ln is diff. by $\ln = \exp^{-1}$, exp diff and ln cont.)
  $\overset{4.9}{\Rightarrow} \ln'(\exp(x)) = \frac{1}{\exp(x)}$ => $\ln'(y) = \frac{1}{y}$ $\forall y \in (0,\infty)$
- let e Euler number "$e^x$"
  If $x = \frac{m}{n}$ with $m \in \mathbb{Z}, n \in \mathbb{Z}^*$, then $e^x = e^{\frac{m}{n}} = (e^m)^{\frac{1}{n}}$ $\left(e = \lim_{n \to \infty} (1 + \frac{1}{n})^n\right)$
  => $\exp(\frac{m}{n}) = e^{\frac{m}{n}}$ (tutorials ana 1: $\exp(1) = e$)

4.24 => Definition: let $a > 0$, $a \in \mathbb{R}$ and $x \in \mathbb{C}$. Then define $a^x = \exp(x \cdot \ln(a))$
=> $a^\cdot : \mathbb{C} \to \mathbb{C}$ (exponential with basis a)

4.25 => proposistion: let $a > 0$
- $a^x = a^{\frac{m}{n}} = \sqrt[n]{a^m}$ for $x = \frac{m}{n}$, $m \in \mathbb{Z}, n \in \mathbb{Z}^*$
- $\forall x, y > 0: \ln(x \cdot y) = \ln(x) + \ln(y)$, $\ln(\frac{1}{x}) = -\ln(x)$ (exp. law)
- $(a^x)^y = a^{xy}$, $x, y \in \mathbb{C}$ and $a^{x+y} = a^x \cdot a^y$
- $\lim_{x \to 0^+} \ln(x) = -\infty$, $\lim_{x \to \infty} = +\infty$, $\ln(1) = 0$, $\ln(e) = 1$
- ln increasing

4.26 => Definition: Sin & Cos

Define $\sin: \mathbb{R} \to \mathbb{R}$, $\cos: \mathbb{R} \to \mathbb{R}$

$\sin_{\mathbb{C}}: \mathbb{C} \to \mathbb{C}$, $\cos_{\mathbb{C}}: \mathbb{C} \to \mathbb{C}$

by $\sin_{\mathbb{C}}(z) = \frac{\exp(iz) - \exp(-iz)}{2i}$, $\cos_{\mathbb{C}}(z) = \frac{\exp(iz) + \exp(iz)}{2}$

$\Rightarrow \sin(x) = \underbrace{\mathrm{Im}(\exp(ix))}_{= \sum_{k=0} \frac{(-1)^k x^{2k+1}}{(2k+1)!}}$ and $\cos(x) = \underbrace{\mathrm{Re}(\exp(ix))}_{= \sum_{k=0} \frac{(-1)^k x^{2k}}{2k!}}$ $\forall x \in \mathbb{R}$

4.27 => proposistion:

- $\sin(0) = 0$, $\cos(0) = 1$
- $\forall x \in \mathbb{R}: e^{ix} = \cos(x) + i \cdot \sin(x)$ (Euler Formula)

  $\Rightarrow \forall n \in \mathbb{N}^*: \underset{(e^{ix})^n}{\underset{\parallel}{e^{inx}}} = \cos(nx) + i \cdot \sin(nx)$

  $= (\cos(x) + i \cdot \sin(x))^n$ (De Moivre)
- $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ (prove by $\mathrm{Re}(e^{i(x+y)}), \ldots$)

  $\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$ $x, y \in \mathbb{R}$
- $\forall x \in \mathbb{R}: \sin(-x) = -\sin(x)$

  $\cos(x) = \cos(-x)$
- For $t \in [0,2]$ $n \in \mathbb{N}^*$: $\frac{t^n}{n!} \geq \frac{t^{n+2}}{(n+2)!}$

  $\overset{IVT}{\underset{\cos \text{ cont}}{\Rightarrow}} \exists$ minimal zero $> 0$ of cos in $[0,2]$

  Call it $x_0$. The real number $2x_0$ is called $\pi$.
- $\sin(t + \frac{\pi}{2}) = \cos(t)$ $\forall t \in \mathbb{R}$, $\cos(t + 2\pi) = \cos(t)$, $\sin(t + 2\pi) = \sin(t)$

  ↳ periodic with period $2\pi$
- $\{x \in \mathbb{R} : \cos(x) = 0\} = \{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$

  $\{x \in \mathbb{R} : \sin(x) = 0\} = \{k\pi : k \in \mathbb{Z}\}$
- $\forall t \in \mathbb{R}: (\sin(t))^2 + (\cos(t))^2 = 1$
- $\lim_{x \to \pm\infty} \frac{\sin(x)}{\cos(x)}$ does not exist by periodicity and points above
- Cos has local maxima at $2\pi k, k \in \mathbb{Z}$, and minima at $(2k+1)\pi, k \in \mathbb{Z}$

  Sin has local maxima at $\frac{1}{2}\pi + 2\pi k$ and minima at $\frac{3\pi}{2} + 2\pi k$, $k \in \mathbb{Z}$

=> Remark: Investigations of the form of proposistion 4.27 are called "curve discussion"

=> Recal: So far we have seen a neccesary condition of local extremum (of differentiable functions)

## 4.4 The theorem by Taylor

=> motivation: - find sufficient conditions for local extrema
- how to "approximate" functions sufficiently differentiable by polynomials $f: I \to \mathbb{R}$

4.28 => **Definition**: Taylor polynomial
Let $I$ be an interval w.m.t.o.p., $y \in I$, $f: I \to \mathbb{R}/\mathbb{C}$, $n$-times differentiable at $y$, where $n \in \mathbb{N}^* \cup \{0\}$. Then the polynomial $T_{n,y}(f)$
$(T_{n,y}(f))(x) := \sum_{k=0}^{n} \frac{(x-y)^k}{k!} \cdot f^{(k)}(y)$ is called Taylor polynomial of $f$ of $n$-th order at $y$.

4.29 => **Theorem**: Taylor's theorem
Let $n \in \mathbb{N}^* \cup \{0\}$ and $I$ interval w.m.t.o.p., $f: I \to \mathbb{R}$ such that $f \in C^n(I;\mathbb{R})$ $n$-times continuously differentiable and $f^{(n)}$ is assumed to be differentiable on the "innerpoints of $I$ ($I$ without its boundary points). Then for every $x \neq y$, $x, y \in I$ $\exists \xi (\min\{x,y\}, \max\{x,y\})$ such that $(R_{n,y}(f))(x) \overset{def}{=} f(x) - (T_{n,y}(f))(x) = \frac{(x-y)^{n+1}}{(n+1)!} f^{n+1}(\xi)$ where $T_{n,y}(f)$ is the Taylor polynomial of $f$ of $n$'th order at $y$. $R_{n,y}(f)$ is called "the rest term" of the Taylor polynomial.

↳ proof: stragegy: define suitable functions $F, G$ and apply MVT

$$F(t) = f(x) - \underbrace{\sum_{k=0}^{n} \frac{(x-t)^k}{k!} f^{(k)}(t)}_{=(T_{n,t}(f))(x)}, \quad G(t) = (x-t)^{n+1}, \quad F/G: I \longrightarrow \mathbb{R}$$

Since $f \in C^n(I;\mathbb{R})$, $F$ is continuous $[\min\{x,y\}, \max\{x,y\}]$ and differentiable $(\min\{x,y\}, \max\{x,y\})$

Since $G \in C^\infty(I;\mathbb{R})$, $G$ is continuous $[\min\{x,y\}, \max\{x,y\}]$ and differentiable $(\min\{x,y\}, \max\{x,y\})$

Also note that $G'(t) = -(n+1)(x-t)^n = 0$ for $t \in (\min\{x,y\}, \max\{x,y\})$

$\overset{MVT}{\underset{gen}{\Rightarrow}} \exists \xi \in (\min\{x,y\}, \max\{x,y\}) : \frac{F(y) - F(x)}{G(y) - G(x)} = \frac{F'(\xi)}{G'(\xi)}$

note that: $F(x) = 0$, $F(y) = f(x) - (T_{n,y}(f))(x)$

$$G(x) = 0, \quad G(y) = (x-y)^{n+1}$$

$$F' = \sum_{k=1}^{n} \frac{k(x-t)^{k-1}}{k!} f^{(k)}(t) - \sum_{k=0}^{n} \frac{(x-t)^k}{k!} f^{(k+1)}(t)$$

$$= \sum_{k=1}^{n} \frac{(x-t)^{k-1}}{(k-1)!} f^{(k)}(t) - \sum_{k=0}^{n} \frac{(x-t)^k}{k!} f^{(k+1)}(t) = -\frac{(x-t)^n}{n!} f^{(n+1)}(t)$$

$$\Rightarrow \left.\begin{aligned} \frac{F(y)-F(x)}{G(y)-G(x)} &= \frac{f(x) - (T_{n,y}(f))(x)}{(x-y)^{n+1}} \\ \frac{F'(\xi)}{G'(\xi)} &= \frac{-\frac{(x-\xi)^n}{n!} f^{n+1}(\xi)}{-(n+1)(x-\xi)^n} \end{aligned}\right\} \frac{f(x) - (T_{n,y}(f))(x)}{(x-y)^{n+1}} = \frac{-\frac{(x-\xi)^n}{n!} f^{n+1}(\xi)}{-(n+1)(x-\xi)^n} \Rightarrow \frac{F(y)-F(x)}{G(y)-G(x)} = \frac{F'(\xi)}{G'(\xi)}$$

4.30 => Remarks:

- By choosing $G$ differently in the above proof we can find other representations of $(R_{n,y}(f))(x)$ (the form in 4.29 is called the Lagrange form)
  e.g. $G(t)=(x-t)^p, p\in\mathbb{N}^* \Rightarrow R_n(x)=\frac{f^{(n)}(\xi)}{n!\,p}(x-y)^p(x-\xi)^{n-p+1}$
- If we add the assumption $f^{(n+1)}(z)=0\ \forall z\in I$ to our 4.29
  $\Rightarrow f(x)=(T_{n,y}(f))(x)\ \forall x\in I$
  This proves that the polynomials of degree $<n+1$ are the "only" solutions to $f^{(n+1)}(x)=0\ \forall x\in I$
- Recall that $\exp(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}$ was in $C^\infty(\mathbb{R};\mathbb{R})$
  let's apply 4.29 with $n\in\mathbb{N}^*$ at $y=0$
  $(T_{n,0}(\exp))(x)=\sum_{k=0}^{n}\frac{(x-0)^k}{k!}\underbrace{\exp^{(k)}(=0)}_{\exp(0)=1}$
  $\exists\, \xi\in(0,x)\cup(x,0)$
  $\Rightarrow (R_{n,0}(\exp))(x)=\sum_{k=n+1}^{\infty}\frac{x^k}{k!}=\frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\xi)$

=> Collorary: let $m\in\mathbb{N}^*$, $m>1$, $a<b\in\mathbb{R}$, $f(a,b)\to\mathbb{R}$, $m$-times differentiable, $f^{(m)}$ continuous at some $x\in(a,b)$
If $f'(x)=f''(x)=\dots=f^{(m-1)}=0$ and $f^{(m)}\neq 0$ then
- if $m$ is even: $f$ has local extremum and it's a local maximum if $f^{(m)}(x)<0$ and a minimum if $f^{(m)}(x)>0$
- if $m$ is odd: $f$ has not a local extremum

↳ proof: use Taylor, use $m=n+1$ and $y=x$ $(t=x)$

## 4.5 Primitives / Stammfunktions

4.32 => Definition: let $I$ be an interval w.m.t.o.p. and $f: I\to\mathbb{R}(\mathbb{C})$. Then a function $S_f: I\to\mathbb{R}(\mathbb{C})$ is called a primitive of $f$ if $S_f'=f$.
(and we say that $f$ "has a primitive/anti-derivative" if a primitive for $f$ exists)

=> Remark: It is not trivial whether a function has a primitive. Moreover, we could define a primitive for functions defined on unions of intervals, but...

4.33 => proposition: Let $I$ be an interval w.m.t.o.p. and $f, g : I \to \mathbb{R}(\mathbb{C})$ with primitives $Sf$ and $Sg$.

1) $\forall C \in \mathbb{R}(\mathbb{C})$: $Sf + C$ is a primitive of $f$

If $\tilde{S}f$ is a primitive of $f$, then $Sf - \tilde{S}f$ is constant on $I$

2) $\forall \lambda \in \mathbb{R}(\mathbb{C})$ $\lambda \cdot Sf + Sg$ is a primitive of $\lambda f + g$

3) Integration by parts: If $f, g$ are differentiable and $Sf'g$ is a primitive of $f'g$, then $f \cdot g - Sf'g$ is a primitive of $fg'$

4) Substitution rule: If $h : J \to I$, where $J$ interval w.m.t.o.p and differentiable, then $Sf \circ h$ is a primitive of $(f \circ h) \cdot h'$

↳ proof:

1) $(Sf + C)' = Sf' + 0 \overset{\text{ass}}{=} f \quad \Rightarrow Sf + C$ primitive of $f$

If $\tilde{S}f$ is a primitive of $f \Rightarrow (Sf - \tilde{S}f)' = f - f = 0$
$\overset{4.18}{\Rightarrow} Sf - \tilde{S}f$ is constant

2) $(\lambda f + g)' = (\lambda Sf)' + Sg' = \lambda f + g$

3) Product rule: $(f \cdot g - Sf'g)' = f'g + fg' - f'g = fg'$

4) Chain rule: $(Sf \circ h)' = (Sf' \circ h) \cdot h' = (f \circ h) \cdot h'$

=> Remark: If we use —at the moment notation— of $\int f$ for $\{Sf + C : C \in \mathbb{R}(\mathbb{C}), Sf \text{ primitive of } f\}$ then items 3 and 4 from 4.33 look like "the integration by parts" formula and substitution rule formula.

4.34 => Example:

- $I = \mathbb{R}$, $f(x) = x^n$ $n \in \mathbb{N}^* \cup \{0\}$
  $Sf = \int f = \{x \mapsto \frac{1}{n+1} x^{n+1} + C, C \in \mathbb{R}(\mathbb{C})\}$
- $I = (0, \infty)$, $f(x) = x^n$, $n \in \mathbb{Z} \setminus (\mathbb{N}^* \cup \{0\})$
  $Sf(x) = \begin{cases} \ln(x), & n = -1 \\ \frac{1}{n+1} x^{n+1}, & n < -1 \end{cases}$
- $I = (-\infty, 0)$, $f(x) = \frac{1}{x} \Rightarrow Sf(x) = \ln|x|$
- $I = (0, \infty)$, $f(x) = \ln(x) = \int \ln(x) = \underbrace{1}_{g} \cdot \underbrace{\ln(x)}_{f} \overset{4.33(3)}{=} x \ln(x) - \int x \cdot \frac{1}{x} + C = x(\ln(x) - 1) + C$

- $I = \mathbb{R}$: $S\exp = \exp$, $S\sin = -\cos$, $S\cos = \sin$
- $n \in \mathbb{N}^*$: $\int \cos^n(x) \overset{n>1}{=} \int \overbrace{\cos(x)}^{g} \cdot \overbrace{\cos^{n-1}(x)}^{f} \overset{\text{int by parts}}{=} \sin(x)\cos^{n-1}(x) - \int \sin(x)(n-1)\cos^{n-2}(x) \sin(x)$
  $\overset{4.33(2)}{=} \sin(x)\cos^{n-1}(x) + (n-1) \int \overbrace{\sin^2(x)}^{= 1 - \cos^2(x)} \cos^{n-2}(x)$
  $= \sin(x)\cos^{n-1}(x) - (n-1) \int \cos^n(x) + (n-1) \int \cos^{n-2}(x)$
  $\Rightarrow \int \cos^n(x) = \frac{1}{n} \sin(x)\cos^{n-1}(x) + \frac{n-1}{n} \int \cos^{n-2}(x)$
  $\overset{n=2}{\Rightarrow} \int \cos^2(x) = \frac{1}{2} \sin(x)\cos(x) + \frac{1}{2} \int 1 = \frac{1}{2}(\sin(x)\cos(x) + x) + C \quad C \in \mathbb{R}(\mathbb{C})$

- $\int \frac{a}{x-b}$ for $f(x) = \frac{a}{x-b}$ on $(b,\infty)$
  use substitution (4.33(4)):
  $h(x) = x-b$, $J=(b,\infty) \Rightarrow h'(x) = 1$, $\tilde{f} = \frac{a}{x}$, $I = (0,\infty)$
  $(\tilde{f} \circ h)(x)h'(x) = \frac{a}{x-b}$
  Since $\int \tilde{f} = a\ln(x)$ we get that $\int \frac{a}{x-b} = a\ln(x-b)$
- $\int \frac{1}{1+x^2} = \tan^{-1}(x) + c = \arctan + c$
  To show $(\tan^{-1}(x))' = \frac{1}{1+x^2}$
- $\int \frac{x}{1+x^2}$ use substitution
  $h(x) = x^2 \Rightarrow h'(x) = 2x$
  $\frac{x}{1+x^2} = \frac{\frac{1}{2}h'(x)}{1+h(x)} = (\tilde{f} \circ h)(x) \cdot h'(x)$, $\tilde{f}(y) = \frac{1/2}{1+y}$
  $\Rightarrow \int \frac{x}{1+x^2} = \frac{1}{2}\ln(1+x^2) + c$, $c \in \mathbb{R}$
- $I = (-1,1)$, $\int \frac{1}{\sqrt{1-y^2}} = \arcsin(y) + c = (\sin|_{(-\pi/2,\pi/2)})^{-1}(y) + c$
  Substitute $y = \sin(x)$
- $\int \frac{1}{\sqrt{x^2+1}} = (\sinh)^{-1}(x) + c = \ln(x + \sqrt{x^2+1})$, $I = \mathbb{R}$ *
- $\int \frac{1}{\sqrt{x^2-1}} = (\cosh)^{-1}(x) + c = \ln(x + \sqrt{x^2-1})$, $I = (1,\infty)$
  Substitute $y = \cosh(x)$

* Substitute $y = \sinh(x)$

$\tan(x) = \frac{\sin(x)}{\cos(x)}$
$\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ bij.
$\Rightarrow \tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

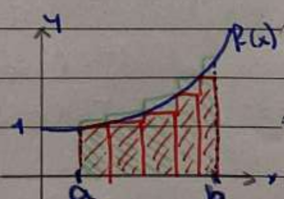
$\sinh(x) = \frac{e^x - e^{-x}}{2}$
$\cosh(x) = \frac{e^x + e^{-x}}{2}$

# Chapter 5: The Riemann Integral

$\Rightarrow$ Motivation: • we would like to define "areas underneath functions"
• what is the "inverse operation" of differentiation (anti-derivative, primitive)

## Section 1: The Darboux-integral

$\Rightarrow$ The idea: we want to have our area of over $[a,b]$ between /// and \\\\\\




5.1 $\Rightarrow$ **Definition:** let $a<b$ real numbers. A finite subset of the interval $[a,b]$ is called partition of the interval if it contains $a$ and $b$.
Convention: If $P$ is a partition, then we write its elements using the ordering of the real numbers. That means if $P$ has $N+1 \in \mathbb{N}^*$ elements then we write
$P = \{t_0, t_1, \dots, t_N\}$ with $t_0 = a < t_1 < \dots < t_N = b$

5.2 => **Definition:** Lower and upper sums

For $a<b$ and a partition $P$ of $[a,b]$, and a function $f:[a,b]\to\mathbb{R}$ bounded, we define

$$L_P(f)=\sum_{i=1}^{N}(t_i-t_{i-1})\cdot\inf\{f(t):t\in[t_{i-1},t_i]\}$$

$$U_P(f)=\sum_{i=1}^{N}(t_i-t_{i-1})\cdot\sup\{f(t):t\in[t_{i-1},t_i]\}$$

where $N+1$ is the number of elements in $P=\{t_i: i=0,\dots,N\}$

$L_P(f)$ is called a lower sum of $f$, $U_P(f)$ is called an upper sum of $f$

5.3 => **Definition:** Integrability

Let $a<b$, $f:[a,b]\to\mathbb{R}$, bounded

Then $f$ is called integrable if the lower integral $\underline{\int}_{[a,b]} f := \sup\{L_P(f): P \text{ partition of } [a,b]\}$ is equal to the upper integral $\overline{\int}_{[a,b]} f := \inf\{U_P(f): P \text{ partition of } [a,b]\}$

Thus $\underline{\int}_{[a,b]} f = \overline{\int}_{[a,b]} = \int_a^b f(x)\,dx$

5.4 => Facts:

- Since $L_P(f)=\sum_{i=1}^{N}(t_i-t_{i-1})\inf_{t\in[t_{i-1},t_i]}f(t)\geq\sum_{i=1}^{N}(t_i-t_{i-1})\inf_{t\in[a,b]}f(t)$
  $=\inf_{t\in[a,b]}f(t)\cdot(b-a)$

  $\wedge$

  $U_P(f)=\sum_{i=1}^{N}(t_i-t_{i-1})\sup_{t\in[t_{i-1},t_i]}f(t)\leq\sup_{t\in[a,b]}f(t)(b-a)$

  $\Rightarrow \overline{\int}_{[a,b]} f$, $\underline{\int}_{[a,b]} f$ are in $\mathbb{R}$
- If $P$ is a partition and $f:[a,b]\to\mathbb{R}$ is constant on every interval $(t_{i-1},t_i)$ for $i=0,\dots,N$ and $P=\{t_i: i=0,\dots,N\}$ then $f$ is integrable and
  $\int_a^b f(x)dx=\underline{\int}_{[a,b]} f=\overline{\int}_{[a,b]} f=\sum_{i=1}^{N}(t_i-t_{i-1})\underbrace{f\left(\tfrac{t_{i-1}+t_i}{2}\right)}_{=c_i}$

  Such functions are called step functions
- $f:[a,b]\to\mathbb{R}$ bounded, integrable $\Leftrightarrow$

  $\forall\varepsilon>0\ \exists P$ partition such that $U_P(f)-L_P(f)<\varepsilon$

  Since: $\forall\varepsilon>0\ \exists P$ partition s.t. $\underline{\int}_{[a,b]} f-L_P(f)<\varepsilon \wedge U_P(f)-\overline{\int}_{[a,b]} f<\varepsilon$

  (that one such $P$ exists follows by taking the union)

  This characterisation is typically used to show integrability by definition
- $f:[a,b]\to\mathbb{R}$, bounded, is integrable $\Leftrightarrow$

  $\exists$ sequence of step functions $g_n$, $h_n$ such that $g_n(x)\leq f(x)\leq h_n(x)$ $\forall x\in[a,b]$ and $\lim_{n\to\infty}\int_a^b g_n(x)\,dx=\lim_{n\to\infty}\int_a^b h_n(x)\,dx$, this limit then equals $\int_a^b f(x)dx$

5.5 => proposistion: properties of the integral

Let $a<b$ real numbers, $f,g:[a,b]\to\mathbb{R}$ bounded

1) If $f,g:[a,b]\to\mathbb{R}$ are integrable and $\lambda$ real number then $f+\lambda g$ integrable and $\int_a^b (f+\lambda g)(x)\,dx = \int_a^b f(x)dx + \lambda\cdot\int_a^b g(x)dx$ ($f\mapsto \int_a^b f(x)dx$ is linear)

2) If $f,g$ are integrable and $f(x)\le g(x)$ $\forall x\in[a,b]$ then $\int_a^b f(x)\,dx \le \int_a^b g(x)dx$ (monotonicity)

3) If $f,g$ integrable $\Rightarrow$ $\left.\begin{array}{l} f_+ := \max\{f,0\} \\ f_- := -\min\{f,0\}\end{array}\right\} \Rightarrow \begin{array}{l} f = f_+ - f_- \\ |f| = f_+ - f_-\end{array}$ integrable

and thus $|f|$ is integrable and $\underbrace{\left|\int_a^b f(x)dx\right|}_{\int_a^b f_+ - \int_a^b f_-} \le \int_a^b |f(x)|dx$

"triangle inequality"

4) If $f,g$ are integrable, then $f\cdot g$ integrable ...

and if $\exists C>0$ $\forall x\in[a,b]$ $g(x)>C$ then also $\frac{f}{g}$ integrable

=> Recall: $f:[a,b]\to\mathbb{R}$, $a<b$ real numbers, bounded

5.6 => Examples:

a) $f:[0,1]\to\mathbb{R}$, $x\mapsto \begin{cases} 1 & x \text{ rational } (x\in\mathbb{Q}\cap[0,1]) \\ 0 & x \text{ irrational } (x\in[0,1]\setminus\mathbb{Q})\end{cases}$ Dirichlet function

Claim: $f$ is not integrable

To do so, we show that $\overline{\int_{[0,1]}} f \neq \underline{\int_{[0,1]}} f$

$\overline{\int_{[0,1]}} f = \inf\{U_p(f) : p \text{ partition of } [0,1]\}$

$= 1$ (by Archimedean principle $\forall i\ \exists r\in\mathbb{Q}\ r\in[t_{i-1},t_i]$)

$\underline{\int_{[0,1]}} f = \sup\{L_p(f) : p \text{ partition of } [0,1]\}$

$= 0$

b) Thomae's function

$g:[0,1]\to\mathbb{R}$, $x\mapsto \begin{cases} \frac{1}{q} & \text{if } x=\frac{p}{q} \text{ for } p\in\mathbb{Z}, q\in\mathbb{N}^*, p,q \text{ coprime} \Leftrightarrow \text{canceld fraction} \\ 0 & \text{if } x \text{ irrational} \wedge x\in[0,1]\end{cases}$

$g$ is integrable (note that $0\le g\le f$)

=> Recall: Let $[a,b]$ interval $a<b$. A function $f:[a,b]\to\mathbb{R}$ is called a step function if $\exists p$ partition of $[a,b]$ such that $f|_{(t_{i-1},t_i)}$ constant $\forall i$ where $p=\{t_i : i=0,\dots,N\}$

$\Leftrightarrow$ step functions are bounded ($f_n:[a,b]\to\mathbb{R}$)

$\Rightarrow$ Recall: $(f_n)_{n\in\mathbb{N}^*}$ converges uniformly to $f:[a,b]\to\mathbb{R}$

$\Leftrightarrow \|f_n - f\|_\infty \overset{\text{def}}{=} \sup_{t\in[a,b]} |f_n(t) - f(t)| \xrightarrow{n\to\infty} 0$

"step functions are the building blocks of the Riemann/Darboux integral"

$\Rightarrow$ Idea: $\{f:[a,b]\to\mathbb{R} : \exists (f_n)_{n\in\mathbb{N}^*}, f_n:[a,b]\to\mathbb{R}$ step function $\forall n\in\mathbb{N}^*$ $f_n \xrightarrow{n\to\infty} f$ uniformly$\} =: Reg([a,b])$ regulated functions on $[a,b]$

5.7 $\Rightarrow$ Theorem: Regulated functions are integrable

Let $a<b$ real numbers, then every $f\in Reg([a,b])$ is integrable.

In particular, continuous functions on $[a,b]$ are integrable.

(any continuous function on $[a,b]$ is in $Reg([a,b])$)

↳ proof:

For the first statement:

Let $f\in Reg([a,b]) \Rightarrow \exists (f_n)$ step functions: $\|f_n - f\|_\infty \xrightarrow{n\to\infty} 0$

By fact 5.4: $f$ integrable $\Leftrightarrow \forall \varepsilon>0\ \exists g,h$ step functions:

$g\le f\le h \wedge \int_a^b h(x)dx - \int_a^b g(x)dx < \varepsilon$

$\Leftrightarrow \exists (g_n),(h_n)$ step functions: $g_n\le f\le h_n \wedge \int_a^b h_n(x)dx - \int_a^b g_n(x)dx \xrightarrow{n\to\infty} 0$

Given $\varepsilon>0$

1) Choose $n\in\mathbb{N}^*$: $\|f_n - f\|_\infty = \sup_{t\in[a,b]} |f_n(t) - f(t)| < \frac{\varepsilon}{\int_a^b 1\,dx}$

2) Choose $g_n, h_n$: Take the partition $p$ of the step function $f_n$

define $h_n(t) := \sup_{x\in(t_{i-1},t_i)} f(x)$, $g_n(t) = \sup_{x\in(t_{i-1},t_i)} f(x)$ where $t\in(t_{i-1},t_i)$ $i=1,\dots,N$

$h_n(t_i) \overset{\text{def}}{=} g_n(t_i) \overset{\text{def}}{=} f(t_i)$ $\forall i=0,\dots,N$

$\Rightarrow g_n \le f\le h_n$

also $h_n(x) - g_n(x) \le |h_n(x) - g_n(x)| \overset{\Delta}{\le} \overbrace{|h_n(x) - f_n(x)|}^{=\sup_{t\in(t_{i-1},t_i)} |f(t) - f_n(x)|} + |f_n(x) - g_n(x)| \le 2\frac{\varepsilon}{\int_a^b 1\,dx}$

by the fact that $f_n$ is constant on $(t_{i-1},t_i)$, $f_n(t_{i-1}) = f(t_{i-1})$ and step 1

$\overset{5.5(2)}{\Rightarrow} \int_a^b h_n(x) - g_n(x)\,dx \le \int_a^b 2\frac{\varepsilon}{\int_a^b 1\,dy}dx \overset{5.5(1)}{=} 2\cdot\frac{\varepsilon}{\int_a^b 1\,dy}\cdot\int_a^b 1\,dx = 2\varepsilon$

$\Rightarrow f$ integrable

The second statement: if $f$ continuous on $[a,b] \overset{3.18}{\Rightarrow} f$ is uniformly continuous on $[a,b]$ ($[a,b]$ compact)

$\forall\varepsilon>0\ \exists\delta>0: \forall x,y\in[a,b]: (x-y)<\delta \Rightarrow |f(x)-f(y)|<\varepsilon)$

We want: for given $\varepsilon>0$ a step function $f_\varepsilon$ s.t. $\|f - f_\varepsilon\|_\infty < \varepsilon$

Take $\delta$ from above and define $P=\{a, a+\delta, a+2\delta, \dots, a+(N-1)\delta, b\}$ for unique $N\in\mathbb{N}^*$ and define $f_\varepsilon(t) = \frac{f(t_{i-1},t_i)}{2}$ for $t\in(t_{i-1},t_i)$ $f_\varepsilon(t_i) = f(t_i)$

S.8 => Proposition: Characteristics of Reg([a,b])

A function $f:[a,b]\to\mathbb{R}$ is in Reg([a,b])

$\Leftrightarrow \forall x\in(a,b)$ the left and the right limit of $f(t)$ as $t\to x$ exists and for $x=a$, $x=b$ the right and left limit of $f(t)$ exists as $t\to a^+$ or $t\to b^-$.

↳ proof: tutorial / lecture notes

S.9 => Examples

c) Using prop. S.8, we can show that Thomae's function (see S.6) is in Reg([0,1]). Likewise one can conclude that the Dirichlet function is not in Reg([a,b]) (S.7 + Dirichlet function not integrable)

↳ see tutorial

d) Any function $f:[a,b]\to\mathbb{R}$ which is continuous up to finitely many points, such that the left and right limit exist at those points, is integrable.

S.10 => Facts:

- For $c\in[a,b]$: $f[a,b]\to\mathbb{R}$ is integrable
  $\Leftrightarrow f|_{[a,c]}:[a,c]\to\mathbb{R}$ and $f|_{[c,b]}:[c,b]\to\mathbb{R}$ are integrable and $\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- If $f:[a,b]\to\mathbb{R}$ integrable, then for any $c\in[a,b]$ $y\in\mathbb{R}$
  $\tilde{f}[a,b]\to\mathbb{R}$, $\tilde{f}(x)=\begin{cases} f(x) & \text{for } x\neq c,\ x\in[a,b] \\ y & \text{for } x=c\end{cases}$ is integrable

## S.2 The fundamental theorem of calculus

Theorem ~~FTC~~

? S.11 => Let $f:[a,b]\to\mathbb{R}$ be integrable. Then the function $F:[a,b]\to\mathbb{R}$ given by $F(x)=\int_a^x f(y)dy$ is continuous. If additionally $f$ is continuous at $x_0\in[a,b]$, then $F$ is differentiable at $x_0$ and $F'(x_0)=f(x)$

↳ proof: 1) F continuous, F is well defined because f integrable on $[a,b] \Rightarrow f|_{[a,x]}:[a,x]\to\mathbb{R}$ is integrable

We show that $\lim_{y\to x^-}F(y)=\lim_{y\to x^+}F(y)=F(x)$ $\forall x\in[a,b]$

Let $h>0$, $x\in[a,b)$ and such that $x+h\leq b$

$F(\underbrace{x+h}_{y})-F(x)\overset{\text{def}}{=}\int_a^{x+h}f(y)dy-\int_a^x f(y)dy=\int_x^{x+h}f(y)dy$

$\Rightarrow |F(x+h)-F(x)| = \left|\int_x^{x+h} f(y)dy\right| \overset{5.4(2)}{\leq} \int_x^{x+h} |f(y)|dy \leq \sup_{y\in[x,x+h]} |f(y)|$

Since $f:[a,b]\to\mathbb{R}$ is bounded

$\overset{5.4(2)}{\Rightarrow} |F(x+h)-F(x)| \leq \sup_{y\in[x,x+h]} |f(y)| \cdot \underbrace{\int_x^{x+h} 1\, dy}_{x+h-x=h} \leq \sup_{y\in[a,b]} |f(y)|\cdot h$

$\overset{\text{sandwich}}{\Rightarrow} \lim_{h\to 0^+} |F(x+h)-F(x)| = 0$

Similarly $\lim_{h\to 0^+} |F(x-h)-F(x)| = 0$

$\Rightarrow F$ is continuous at $x$ $\forall x\in[a,b]$

2) If $f$ continuous at $x_0$, then to show that $F$ is differentiable and $F'(x_0) = f(x_0)$

As in (1) first let $h>0$, $x_0\in[a,b)$ such that $x_0+h\leq b$

$\frac{F(x_0+h)-F(x_0)}{h} - f(x_0) = \frac{1}{h}\int_{x_0}^{x_0+h} f(y)dy - \frac{1}{h}\int_{x_0}^{x_0+h} f(x_0)dy$

$\overset{\text{lin}}{=} \frac{1}{h}\int_{x_0}^{x_0+h} (f(y)-f(x_0))dy$

$\overset{\Delta\text{ineq.}}{\Rightarrow} \left|\frac{F(x_0+h)-F(x_0)}{h} - f(x_0)\right| \leq \frac{1}{h}\int_{x_0}^{x_0+h} |f(y)-f(x_0)|dy$

$\leq \frac{1}{h} \sup_{y\in[x_0,x_0+h]} |f(y)-f(x_0)|\cdot h$

$= \frac{1}{h} \sup_{y\in[x_0,x_0+h]} |f(y)-f(x_0)|$

$\overset{f\text{ cont.}}{\Rightarrow} \lim_{h\to 0^+} \sup_{y\in[x,x_0]} |f(y)-f(x_0)| = 0$

Similarly $\lim_{h\to 0^+} |F(x_0-h)-F(x_0)-f(x_0)| = 0$

$\Rightarrow F$ is differentiable at $x_0$ with $F'(x_0) = f(x_0)$

5.12 $\Rightarrow$ Corollary: let $a<b$ real, $f:[a,b]\to\mathbb{R}$ continuous. Then $f$ has a primitive/anti-derivative $F$ given in theorem 5.11.
Conversely, for any primitive $G$ of $f$ on $[a,b]$ it holds that
$\int_a^b f(y)dy = G(b)-G(a)$

5.13 $\Rightarrow$ Corollary: let $a<b$ real, $f:[a,b]\to\mathbb{R}$

1) $H(x) = \int_x^c f(y)dy$, $x\in[a,c]$, $c\in[a,b]$
$H:[a,c]\to\mathbb{R}$. If $f$ continuous on $[a,b]$
$\Rightarrow H'(x) = -f(x)$ (since $\int_x^c f(y)dy = \int_a^c f(y)dy - \int_a^x f(y)dy$)

2) $g:[a,b]\to\mathbb{R}$, $f,g$ continuously differentiable
Then $\int_a^b f(x)g'(x)dx = \underbrace{f\cdot g|_a^b}_{=f(b)g(b)-f(a)g(a)} - \int_a^b f'(x)g(x)dx$ "integration by parts"

3) Substitution rule
$h:[\alpha,\beta]\to\mathbb{R}$ continuously differentiable, $h([\alpha,\beta])\subseteq[a,b]$,
$f$ continuous
$\int_\alpha^\beta f(h(y))\cdot h'(y)dy = \int_{h(\alpha)}^{h(\beta)} f(x)dx$

$\Rightarrow$ Convention: $\int_a^b f(y)dy = -\int_b^a f(y)dy$ for any $a,b\in\mathbb{R}$

=> Remark: For applying the substitution rule you must argue that it is applicable but you may use
" $\int_1^2 f(x^2)\,dx = \left| \begin{smallmatrix} y = x^2 \\ dy = 2x\,dx \end{smallmatrix} \right| = \int_{h^{-1}(1)}^{h^{-1}(2)} f(y)\cdot\frac{1}{2\sqrt{y}}\,dy$ "

5.14 => Example: $\int_1^3 \ln(x)\,dx = x\ln(x) - 1 \Big|_{x=1}^{x=3} = 3(\ln(3)-1)+1$

5.15 => Corollary: let $a<b$ real, $f:[a,b]\to\mathbb{R}$ continuous
Suppose $f(x)\geq 0 \;\forall x\in[a,b]$
Then $\int_a^b f(x)\,dx = 0 \Rightarrow f=0 \quad (f(x)=0 \;\forall x\in[a,b])$
↳ proof: let $F(x) = \int_a^x f(y)\,dy$, $F(a)=F(b)=0$ and
$F$ differentiable by FTC
Since $F'(x) = f(x)\geq 0 \;\forall x\in[a,b]$, $F$ nondecreasing
$\Rightarrow 0 = F(a) \leq F(x) \leq F(b) = 0$
$\Rightarrow F(0)$

## 5.3 Improper Riemann-integrals

=> So far $f$ integrable => $f$ bounded (by def)

5.16 => Definition: let $-\infty < a < b \leq +\infty$ and $f[a,b)\to\mathbb{R}$ such that $f|_{[a,c]}$ is integrable for any $c\in[a,b)$. Then $f$ is called improperly Riemann integrable if $\lim_{c\to b^-}\int_a^c f(y)\,dy$ exists in $\mathbb{R}$
analogously if $f:(a,b]\to\mathbb{R}$ with $f|_{[c,b]}$ integrable $\forall c\in(a,b]$ then we call $f$ improperly integrable if $\lim_{c\to a^+}\int_c^b f(y)\,dy$ exists in $\mathbb{R}$
We say that $f: I\to\mathbb{R}$ is absolutely (improperly) integrable if $|f|: I\to\mathbb{R}$ is (improperly) integrable

5.17 => Facts:
- Recall $\lim_{c\to b^-}\int_b^c f(y)\,dy$ exists in $\mathbb{R} \Leftrightarrow \forall$ seq $(c_n)_{n\in\mathbb{N}}$ in $[a,b)$ converging to $b$ we have that $\lim_{n\to\infty}\int_a^{c} f(y)\,dy$ exists and is independent of $(c_n)$
- If $f:[a,b]\to\mathbb{R}$ is integrable, then $f|_{[a,b)}$ is improperly integrable and $\int_a^b f(y)\,dy = \lim_{c\to b^-}\int_a^c f(x)\,dx$ (by FTC, which implied that $c\mapsto\int_a^c f(x)\,dx$ is continuous)
- We have seen that $f:[a,b]\to$ integrable $\Rightarrow |f|:[a,b]\to\mathbb{R}$ is integrable (5.4(3))

5.18 => Example: $f:[0,\infty)\to\mathbb{R}$, $f(x)=\begin{cases}\frac{\sin(\pi x)}{x}, & x>0\\ \pi, & x=0\end{cases}$

$f$ improperly integrable?

note that $f|[a,b]$ is integrable $\forall b>0$, because $f|[0,b]$ is continuous ($\lim_{x\to 0^+}\frac{\sin(\pi x)}{x}=\pi$)

We need to check if $f|[b,\infty)$ is improperly integrable for some $b>0$

Let $b=1$, $c>1$, $\int_1^c f(x)\,dx=\int_1^c\frac{\sin(\pi x)}{x}\,dx$ ($\lfloor c\rfloor$: floor function, $c=\lfloor c\rfloor+r_c$, $r_c\in[0,1)$, $\lfloor c\rfloor\in\mathbb{N}^*$)

$=\int_1^{\lfloor c\rfloor}\frac{\sin(\pi x)}{x}\,dx+\int_{\lfloor c\rfloor}^c\frac{\sin(\pi x)}{x}\,dx$

Note that the sign of $\sin(\pi x)$ is $\begin{cases}+ & \text{if } x\in(n,n+1)\ n \text{ even}\\ - & \text{if } x\in(n,n+1)\ n \text{ odd}\end{cases}$

$\int_n^{n+1}\frac{\sin(\pi x)}{x}\,dx\begin{cases}\geq 0, & n \text{ even}\\ \leq 0, & n \text{ odd}\end{cases}$

$\int_1^{\lfloor c\rfloor}f(x)\,dx=\sum_n^{n+1}\underbrace{\int_n^{n+1}f(x)\,dx}_{=a_n}$

Use Leibniz (for alternating series) to conclude that $\sum_n^{\lfloor c\rfloor-1}a_n$ converges

For that check that $|a_{n+1}|\leq|a_n|\ \forall n\in\mathbb{N}^*$ and $a_n\xrightarrow{n\to\infty}0$

$\frac{|\sin(\pi(x+1))|}{x+1}\leq\frac{|\sin(\pi x)|}{x}$

$\Rightarrow\int_1^{\lfloor c\rfloor}f(x)\,dx$ converges as $c\to\infty$

$\left|\int_{\lfloor c\rfloor}^c f(x)\,dx\right|=\left|\int_{\lfloor c\rfloor}^c\frac{\sin(\pi x)}{x}\,dx\right|\leq\int_{\lfloor c\rfloor}^{\lfloor c\rfloor+1}\frac{|\sin(\pi x)|}{x}\,dx\leq\frac{1}{\lfloor c\rfloor}\cdot\int_{\lfloor c\rfloor}^{\lfloor c\rfloor+1}1\,dx$

$=\frac{1}{\lfloor c\rfloor}\xrightarrow{c\to\infty}0$

$\Rightarrow\lim_{c\to\infty}\int_1^c f(x)\,dx=\sum_{n=1}^{n+1}a_n\in\mathbb{R}$

$\Rightarrow f$ is improperly integrable on $[0,\infty)$

$f$ absolutely improperly integrable?

$|a_n|=\int_n^{n+1}\frac{|\sin(\pi x)|}{x}\,dx\geq\int_n^{n+1}\frac{|\sin(\pi x)|}{n+1}=\frac{1}{n+1}\int_n^{n+1}|\sin(\pi x)|\,dx$

$=\frac{1}{n+1}\left|\cos(\pi x)\Big|_n^{n+1}\right|=\frac{1}{n+1}|\cos((n+1)\pi)-\cos(n\pi)|=\frac{2}{n+1}$

$\Rightarrow$ Since $\sum_{n=1}^{\infty}|a_n|$ diverges, $|f|$ not impropely integrable

5.19 => proposistion: If $f,g:[a,b]\to\mathbb{R}$ such that $\forall x\in(a,b)$ $f|[a,x]$, $g|[a,x]$ integrable, $-\infty<a<b\leq+\infty$. Then:

1) If $|f(x)|\leq|g(x)|$ $\forall x\in(a,b)$ and $g$ is absolutely improperly integrable, then also $f$ is absolutely improperly integrable

2) If $g$ is absolutely improperly integrable then $g$ is also improperly integrable

↳ proof: tutorial

## 5.4 Sequences of functions and exchanging limits

=> Recall: We studied sequences of functions before.
In particular we studied pointwise and uniform convergence
Let $f_n : [a,b] \to \mathbb{R}$ ($a<b$ real), $n \in \mathbb{N}^*$, $f : [a,b] \to \mathbb{R}$

$$\begin{cases} f_n \to f \text{ pointwise} \overset{\text{def}}{\iff} \forall x \in [a,b] : f_n(x) \xrightarrow{n\to\infty} f(x) \\ f_n \to f \text{ uniformly} \overset{\text{def}}{\iff} \forall \varepsilon > 0 \; \exists N \in \mathbb{N}^* : \forall x \in [a,b] \; \forall n \geq N \; |f_n(x) - f(x)| < \varepsilon \end{cases}$$

For uniform convergence, if $f_n$ continuous then $f$ as well
(don't holds for pointwise convergence)

5.20 => **Theorem:** Let $a<b$ real and $f_n : [a,b] \to \mathbb{R}$ (c), $n \in \mathbb{N}^*$,
$f : [a,b] \to \mathbb{R}$ such that
- $f_n$ are integrable $\forall n \in \mathbb{N}^*$
- $f_n \to f$ uniformly

then $f$ is integrable and $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \overbrace{f(x)}^{\lim_{n\to\infty} f_n(x)}\,dx$

↳ proof: Goal: use (facts of) definition of integrability
Let $\varepsilon > 0$ $\exists N \in \mathbb{N}^*$ $\forall n \geq N$ $\sup_{x \in [a,b]} |f(x) - f_n(x)| < \varepsilon$ (uniform convergence)
Fix $n \geq N$ $\exists p$ partition of $[a,b]$ : $U_p(f_n) - L_p(f_n) < \varepsilon$ ($f_n$ integrable)

$$U_p(f) - L_p(f) \overset{\Delta\text{ineq}}{\leq} \underbrace{|U_p(f) - U_p(f_n)|}_{=(b-a)} + \underbrace{|U_p(f_n) - L_p(f_n)|}_{<\varepsilon} + |L_p(f_n) - L_p(f)|$$

$$U_p(f) - U_p(f_n) = \overbrace{\sum_{i=1}^{m} (t_i - t_{i-1})}\overbrace{\left(\sup_{t \in [t_{i-1}, t_i]} f(t) - \sup_{t \in [t_{i-1}, t_i]} f_n(t)\right)} \text{ for } p = \{t_0, \dots, t_m\}_{m \in \mathbb{N}^*}$$

$\Rightarrow U_p(f) - L_p(f) \leq 2(b-a)\varepsilon + \varepsilon =: \tilde{\varepsilon}$
$\Rightarrow f$ is integrable
The second statement follows as for $f_n$ being a step function

5.21 => Examples:

a) $f_n : [0,1] \to \mathbb{R}$, $x \mapsto x^n(1-x)$, $n \in \mathbb{N}^*$
$(f_n)_{n\in\mathbb{N}^*}$ converges uniformly by A: guess the limit, B: cauchy criteria
Fix $x \in [0,1]$: $\lim_{n\to\infty} (x^n - x^{n-1}) = 0$
To show $\sup_{x \in [0,1]} |f_n(x) - 0| \xrightarrow{n\to\infty} 0$ by computing max of $f_n$
By 5.20 $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0$ $(= \int_0^1 0\,dx)$
(note $\int_0^1 f_n(x)\,dx = \int_0^1 x^n - x^{n+1}\,dx = \frac{1}{n+1} - \frac{1}{n+2} \xrightarrow{n\to\infty} 0$)

b) $f_n : [0,1] \to \mathbb{R}$, $x \mapsto x^n$, $n \in \mathbb{N}^*$
$(f_n)_{n\in\mathbb{N}^*}$ does not converges uniformly (because ptw limit is not cont.)

=> Remark: Above means the assumptions on S.20 are not satisfied while the statement still holds (see also tutorial)
However in general the uniform convergence in S.20 can not be dropped.

S.22 => Theorem: Exchange derivatives & limits
Let $f_n : [a,b] \to \mathbb{R}$ ($\mathbb{C}$) be continuously differentiable $\forall n \in \mathbb{N}^*$ and such that a) $\exists x_0 \in [a,b] : f_n(x_0) \xrightarrow{n\to\infty} c \in \mathbb{R}$ and
b) $\exists g : [a,b] \to \mathbb{R} : f_n' \xrightarrow{n\to\infty} g$ uniformly
Then $(f_n)_{n\in\mathbb{N}^*}$ converges uniformly to a limit $f : [a,b] \to \mathbb{R}$ ($\mathbb{C}$) and $f$ is $C^1([a,b])$ (continuously differentiable) and $f' = g$
(Note that this can be formulated as $(\lim_{n\to\infty} f_n)'(t) = \lim_{n\to\infty}(f_n'(t))$)
↳ proof: by S.20 and FTC
note that $g$ is continuous (as uniform limit of cont. functions)
$\overset{FTC}{\Rightarrow} f(x) = \int_{x_0}^{x} g(y)dy + c$ defines a differentiable function $f : [a,b] \to \mathbb{R}$ such that $f' = g$
$\overset{FTC}{\Rightarrow} f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(y)dy \xrightarrow[S.20]{n\to\infty} c + \int_{x_0}^{x} g(y)dy \quad \forall x \in [a,b]$
This already implies that $f_n(x)$ converges to $c + f(x)$ $\forall x \in [a,b]$
and for $x = x_0$: $f(x) = c$
$|f(x) - f_n(x)| = |c + \int_{x_0}^{x} g(y)dy - f_n(x_0) - \int_{x_0}^{x} f_n'(y)dy| \le \overbrace{|c - f_n(x_0)|}^{\xrightarrow{n\to\infty} 0} + \overbrace{|\int_{x_0}^{x} g(y) f_n'(y)dy|}^{*}$
$* \le \int_{x_0}^{x} |g(y) - f_n'(y)| dy$
$\le \sup_{y\in[a,b]} |g(y) - f_n'(y)| \xrightarrow{n\to\infty} 0$
$\Rightarrow f_n \xrightarrow{n\to\infty} f$ uniformly, the rest follows from definition of $f$ and FTC

S.23 => Example: $f_n(x) = \frac{\sin(nx)}{n}$ on $[0,1]$ converges uniformly to 0
but $f_n'(x) = \sin(nx)$ does not converge

S.24 => Facts:
- S.22 also holds if $f_n$ is only assumed to be differentiable
- Note that uniform convergence of a sequence of continuous differentiable functions is in general not sufficient to conclude that the limit is differentiable
- Analogous statements as S.20 and S.22 hold for series of functions ($\sum_{k=1}^{\infty} f_k = \lim_{n\to\infty} \sum_{k=1}^{n} f_k$)

5.25 $\Rightarrow$ Example: $\exp: \mathbb{R} \to \mathbb{R},\ x \mapsto \sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges uniformly on every compact interval $[a,b]$

$\overset{5.20}{\Rightarrow} \int_0^1 \exp(y)\,dy = \int_0^1 \sum_{k=0}^{\infty} \frac{y^k}{k!}\,dy = \int_0^1 \lim_{N\to\infty} \sum_{k=0}^{N} \frac{y^k}{k!}\,dy = \lim_{N\to\infty} \sum_{k=0}^{N} \int_0^1 \frac{y^k}{k!}\,dy$

$= \lim_{N\to\infty} \sum_{k=1}^{N} \frac{1}{k!} \frac{1}{k+1} = \lim_{N\to\infty} \sum_{k=1}^{N} \frac{1}{(k+1)!} = \sum_{k=1}^{\infty} \frac{1}{k!} = e - 1 = \exp\big|_0^1$

Similarly one can differentiate exp

to apply 5.22 check if $\sum_{k=1}^{\infty} k \cdot \frac{x^{k-1}}{k!} \left(= \sum_{k=1}^{\infty} \frac{d}{dx} \frac{x^k}{k!}\right)$ is uniform convergent

$\sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = \exp(x)$ converge uniformly

$\overset{5.22}{\Rightarrow} \exp'(x) = \sum_{k=0}^{\infty} \frac{d}{dx}\left(\frac{x^k}{k!}\right) = \exp(x)$

## 5.5 Power series

$\Rightarrow$ **Definition:** Given $(a_k)_{k\in\mathbb{N}_0}$ and $z_0 \in \mathbb{R}$ we call the "series" of function $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ a power series centered at $z_0$ with coefficients $(a_k)$. For such power series, we define its radius of convergence $R = \sup\{|z-z_0| : z \in \mathbb{C}/\mathbb{R} : \sum_{k=0}^{\infty} a_k (z-z_0)^k \text{ converges}\}$

$\Rightarrow$ **proposition**: let $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ be a power series. Then

a) $R = \frac{1}{\lim_{k\to\infty} \sup \sqrt[k]{|a_k|}} =: \tilde{R}$, $R \in [0,\infty]$ (convention: $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$)

If $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$ exists, then $R = $ limit

b) $\forall z \in \mathbb{R}: |z-z_0| > R \Rightarrow \sum_{k=0}^{\infty} a_k |z-z_0|^k$ diverges

c) $\forall z \in \mathbb{R}: |z-z_0| < R \Rightarrow \sum_{k=0}^{\infty} a_k |z-z_0|^k$ converges absolutely

d) $\forall r \in (0,R): \sum_{k=0}^{\infty} a_k (z-z_0)^k$ converges uniformly, absolutely on $\{z \in \mathbb{R} : |z-z_0| \le r\}$

e) The function $z \mapsto \sum_{k=0}^{\infty} a_k (z-z_0)^k$ is continuous from $(z_0 - R, z_0 + R)$ to $\mathbb{R}$ (by $k$ as uniform limit of continuous func.)

$\hookrightarrow$ proof: a) To show $R = \tilde{R}$. Suppose $z_0 = 0$. Let $z \in \mathbb{R}: |z| < \tilde{R}$ that is

$|z| < \frac{1}{\lim_{k\to\infty} \sqrt[k]{|a_k|}} \Leftrightarrow \lim_{k\to\infty} \sqrt[k]{|a_k||z|^k} < 1$

$\overset{\text{root test}}{\Rightarrow} \sum_{k=0}^{\infty} a_k (z-0)^k$ converges absolutely $\Rightarrow \tilde{R} \le R$

Suppose $\tilde{R} < R \Rightarrow \exists z \in \mathbb{R}: \tilde{R} < |z|$ and $\sum_{k=0}^{\infty} a_k z^k$ converges

$\Rightarrow \lim_{k\to\infty} \sqrt[k]{|a_k||z|^k} > 1$

$\overset{\text{root test}}{\Rightarrow} \sum_{k=0}^{\infty} a_k z^k$ does not converge $\lightning$

b) follows from (a)

c) consequence of (d), (d) $\Rightarrow$ (c)

d) Let $r \in (0,R)$ for the case $R > 0$

$\overset{\text{def } R}{\Rightarrow} \exists w \in \mathbb{R}: r < |w| < R$ and $\sum_{k=0}^{\infty} a_k w^k$ converges

$\overset{\text{diver. test}}{\Rightarrow} a_k w^k \xrightarrow{k\to\infty} 0 \Rightarrow \exists c > 0: \forall k \in \mathbb{N}: |a_k w^k| \le c$

Let now $z \in \mathbb{R} : |z| \leq r$

$|a_k z^k| = |a_k w^k| \cdot |\frac{z^k}{w^k}| = |a_k w^k| \cdot |\frac{z}{w}|^k \leq c \cdot (\frac{r}{|w|})^k =: M_k$

by Weierstraß test, $\sum M_k < \infty$, $\frac{r}{|w|} < 1 \Rightarrow \sum_{k=0}^{\infty} |a_k z^k|$ converges uniformly

=> Example:

- $\sum_{n=0}^{\infty} \frac{1}{n!} (z-0)^2$ converges for every $z \in \mathbb{R}$
- $\sum_{n=0}^{\infty} z^n$, here: $z_0 = 0$, $a_k = 1$ $\forall n \in \mathbb{N}$, converges $\forall z \in \mathbb{R} : |z| < 1$

5.28 => proposistion: Differentiation of power series

Let $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ be a power series, with radius of convergence $R$. Then the powerseries given by $\sum_{k=0}^{\infty} (k+1) a_{k+1} (z-z_0)^k$ has the same radius of convergence $R$. The function $f : z \mapsto \sum_{k=0}^{\infty} a_k (z-z_0)^k$ is differentiable with $f'(z) = \sum_{k=0}^{\infty} (k+1) a_{k+1} (z-z_0)^k$ for $z \in (z_0-R, z_0+R)$

Moreover, $f \in C^{\infty}(z_0-R, z_0+R)$ with $f^{(m)}(z) = \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} a_{k+m} (z-z_0)^k$

↳ proof: First show that $\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|} = \overline{\lim_{k\to\infty}} \sqrt[k]{(k+1)|a_{k+1}|}$, (hint: by using the characterisation of $\overline{\lim}$ of largest partial limit point (2.2(4))

2) Recall that differentiability is a local property, hence for given $z \in (z_0-R, z_0+R)$ we can restrict ourselfs to a closed, bounded interval inside $(z_0-R, z_0+R)$ containing $z$

by 5.27 (d) and the first point, $\sum_{k=0}^{\infty} (k+1) a_{k+1} (z-z_0)^k$ converges uniformly on such interval. Furthermore, $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ converges at $z = z_0$

$\overset{\text{exch.lim.diff}}{\underset{5.22}{\Rightarrow}}$ $f$ continuously differentiable and $f'(z) = \sum_{k=0}^{\infty} (k+1) a_{k+1} (z-z_0)^k$

3) follows by induction.

5.29 => Remark:

1) In the definition of a power series, we can replace $\mathbb{R}$ by $\mathbb{C}$ (d. for $\mathbb{C}$)

A function $f : D \subseteq \mathbb{R}/\mathbb{C} \to \mathbb{R}/\mathbb{C}$, $D$ is an interval/disc is called (real) analytic if $\forall z_0 \in D$ $\exists (a_k)_{k \in \mathbb{N}}$ an interval/a disc $\subseteq D$: $f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$

2) For a powerseries with radius of convergence $R$, $R > 0$, it is in general not clear what happens for $z = z_0 \pm R$. More precisely, $\sum_{k=0}^{\infty} a_k (\pm R)^k$ may converge absolutely or diverge. We investigate these points by using our facts for series (of real numbers).

For example: $\sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} \overset{5.27}{\Rightarrow} R = 1 \Rightarrow$ converges on $(-1, 1)$ and additionally at $-1$, but not at $1$.

# Chapter 6: Differentiation of functions in several variables

=> motivation: So far $f:(a,b)\to\mathbb{R}$ we define $x\in(a,b)$: $f'(x)$ exists,
now: functions e.g. $f(x,y)=x^2y+e^y$, $f:\mathbb{R}^2\to\mathbb{R}$
here functions $f: D\subseteq\mathbb{R}^p\to\mathbb{R}^m$

## 6.1 Preliminaries (converge in $\mathbb{R}^p$, open sets, linear mappings)

6.1 => Recall: $(\mathbb{R}^p, d_2)$ metric space, where $d_2(x,y)=\|x-y\|_2=\left(\sum_{k=1}^{p}|x_k-y_k|^2\right)^{1/2}$
because $\forall x,y\in\mathbb{R}^p$: $\left|\sum_{i=1}^{p} x_iy_i\right| \leq \underbrace{\|x\|_2}_{\left(\sum_{i=1}^{p}|x_i|^2\right)^{1/2}} \|y\|_2$ (cauchy Schwarz)
"Euclidean norm"

$X$ vector space over $\mathbb{R}$. Then $\|.\|: X\mapsto[0,\infty)$ is called norm if

- $\|x\|=0 \Leftrightarrow x=0 \quad \forall x\in X$
- $\forall x\in X, \forall\lambda\in\mathbb{R}: \|\lambda x\|=|\lambda|\,\|x\|$
- $\forall x,y\in X: \|x+y\|\leq\|x\|+\|y\|$

$\Rightarrow d(x,y)=\|x-y\|$ defines a metric on $X$

$\overset{\text{cauchy Schwarz}}{\Rightarrow}$ $\|.\|_2$ is a norm
$\|x+y\|_2\leq\|x\|_2+\|y\|_2$ (minkowski)
$\Rightarrow d_2$ defines metric

This is all true for general inner product spaces $X$ with
$d(x,y)=\|x-y\|=\sqrt{\langle x-y\rangle,\langle x-y\rangle}$

6.2 => Facts: Convergence of sequences in $(\mathbb{R}^n, d_2)$
Let $(x_n)_{n\in\mathbb{N}^*}$ be a sequence in $\mathbb{R}^p$
e.g. $p=2$ $\begin{pmatrix} e^{-n} \\ 1/n^2 \end{pmatrix}\in\mathbb{R}^2 \ \forall n$
claim: $(x_n)$ converges to $\begin{pmatrix}0\\0\end{pmatrix}$ in $(\mathbb{R}^2,d_2)$
$d_2\left(\begin{pmatrix} e^{-n} \\ 1/n^2 \end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)=\left\|\begin{pmatrix} e^{-n} \\ 1/n^2 \end{pmatrix}\right\|_2=\sqrt{e^{-2n}+\frac{1}{n^4}}\xrightarrow{n\to\infty}0$

1) In the metric (normed) space $(\mathbb{R}^2,d_2)$ a sequence $(x_n)_{n\in\mathbb{N}^*}$ converges to $x\in\mathbb{R}^p$ if and only if $\lim_{n\to\infty} x_{n,i}=x_i \ \forall i=1,\dots,p$ in $(\mathbb{R}^2,d_2)$
$\overset{\text{def}}{\Leftrightarrow} |x_{n,i}-x_i|\xrightarrow{n\to\infty}0$ where denotes the i-th component
because $\forall x\in\mathbb{R}^p$: $\max_{i=1,\dots,p}|x_i|\leq\underbrace{\|x\|_2}_{\left(\sum_{i=1}^{p}|x_i|^2\right)^{1/2}}\leq\sqrt{p}\max_{i=1,\dots,p}|x_i|$
$\left\{\sum_{i=1}^{p}|x_i|\leq\max_{i=1,\dots,p}|x_i|^2\cdot p\right\}$

↳ proof: If $x_n\to x$ in $(\mathbb{R}^p,d_2)$
$\overset{\text{def}}{\Leftrightarrow} \|x_n-x\|_2=0 \ (n\to\infty)$
$\Leftrightarrow \max_{i=1,\dots,p}|x_{n,i}-x_i|\to0 \ (n\to\infty)$
$\Leftrightarrow |x_{n,i}-x_i|\to0 \ (n\to\infty) \ \forall i=1,\dots,p$

2) The previous point shows that a function $f: D \subseteq \mathbb{R}^p \to \mathbb{R}^m$ is continuous with respect to $d_2 \Leftrightarrow \forall$ seq $(x_n)_{n\in\mathbb{N}^*}$ in D converging componentwise to some $x \in D \Rightarrow f(x_n)$ converges componentwise to $f(x)$. Since $f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}$ the later covergence is equivalent to $f_i(x_n) \to f_i(x)$ $\forall i \in \{1, \dots, p\}$ this is why we can often restrict ourselfs to function $f: D \subseteq \mathbb{R}^p \to \mathbb{R}$

6.3 $\Rightarrow$ Examples

a) The functions add: $\mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x+y$ are continuous
mult: $\mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x \cdot y$

b) The function $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 e^y + \arctan(x \cdot y^{10})$ is continuous because it is a composition of continuous functions

c) $f: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \to \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

$f$ is continuous at any point $(x,y) \neq (0,0)$ as composistion of continuous functions. The point $(0,0)$ requires further attention:
$(x_n, y_n)^T \to (0,0) \overset{?}{\Rightarrow} \frac{x_n y_n}{x_n^2 + y_n^2} \to 0 = f(0,0)$ ?
Let $x_n = y_n = \frac{1}{n}$, $n \in \mathbb{N}^*$: $(x_n, y_n) \to (0,0)$, but $\frac{1/n \cdot 1/n}{2/n^2} = \frac{1}{2}$ $\forall n \in \mathbb{N}^*$
$\Rightarrow$ Not continuous

d) $g: \mathbb{R}^2 \to \mathbb{R}$: $(x,y) \to \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = 0 \end{cases}$ $\quad g(x,y) = x \cdot f(x,y)$

Then $g$ is continuous (at $(0,0)$)
$|g(x,y)| = |x| \cdot \left|\frac{xy}{x^2+y^2}\right| \leq |x| \cdot \frac{1}{2} \xrightarrow{(x,y)\to(0,0)} 0$ $\quad \forall (x,y) \in \mathbb{R}^2$
$\Rightarrow |g(x,y)| \to 0$ for $(x,y) \to (0,0)$
$\Rightarrow g$ continuous at $(0,0)$

6.4 $\Rightarrow$ Definition: Bounded linear mapping

Let $(X, \|\cdot\|_X)$ $(Y, \|\cdot\|_Y)$ be normed spaces and $T: X \to Y$ linear. We call $T$ bounded and write $T \in \mathcal{L}(X,Y)$ if $\exists c > 0: \forall x \in X: \|T(x)\|_Y \leq C \cdot \|x\|_X$ *
(the minimal such $C$ is called the "operator norm" of $T$)

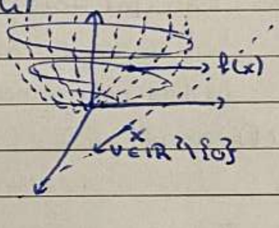
6.5 $\Rightarrow$ Proposistion: Let $X, Y$ be normed spaces, $T: X \to Y$ linear. Then $T$ is bounded if and only if $T$ continuous (w.r.t metrics $d_X(x,y) = \|x-y\|_X$, $d_Y(x,y) = \|x-y\|_Y$)
$\Leftrightarrow T$ is continuous at $0 \in X$
Moreover, if $(X, \|\cdot\|_X) = (\mathbb{R}^p, \|\cdot\|_2)$, then every linear mapping is bounded

↳ proof: Suppose T bounded. let $(x_n)$ in X converging to $x \in X$
There exists a constant c such that (*) holds
$\Rightarrow \|T(x_n)-T(x)\|_Y \overset{T \text{ linear}}{=} \|T(x_n - x)\|_Y \leq C\|x_n - x\| \xrightarrow{n\to\infty} 0$
$\Rightarrow T(x_n) \xrightarrow{n\to\infty} T(x)$ in $(Y, \|\cdot\|_Y)$
If $X = \mathbb{R}^p$ choose some basis of $\mathbb{R}^p$ $(e_i)_{i=1}^{p}$
$x = \sum_{i=1}^{p} x_i e_i$
$\|T(x)\|_Y = \|T(\sum_{i=1}^{p} x_i e_i)\|_Y = \|\sum_{i=1}^{p} x_i T(e_i)\|_Y \overset{\Delta}{\leq} \sum_{i=1}^{p} |x_i| \|T(e_i)\| \leq C\|x\|_2$
for $C = (\sum_{i=1}^{p} \|T(e_i)\|^2)^{1/2}$

[$f: D \subseteq \mathbb{R} \to \mathbb{R}$, $|f(x)-f(y)| \leq C|x-y|$ $\forall x,y \in D$ are called Lipschitz continuous]

## 6.2 Directional derivatives & Differential

6.6 ⇒ **Definition:** $(X,d)$ metric space $((\mathbb{R}^p, d_2) = (\mathbb{R}^p, \|\cdot\|_2))$, $M \subseteq X$, is called open if $\forall x \in M\ \exists r>0: B_r(x) \overset{def}{=} \{y \in X : d(x,y) < r\} \subseteq M$
(Note: $M$ open $\Leftrightarrow$ $X \setminus M$ closed, chap. 4)




⇒ Motivation: $f: \mathbb{R}^2 \to \mathbb{R}$
$(x,y) \mapsto x^2 + y^2$
$f_v: \mathbb{R} \to \mathbb{R}$
$f_v(t) := f(x + t\cdot v)$

6.7 ⇒ **Definition:** Directional derivatives & partial derivatives
let X, Y normed spaces over $\mathbb{R}$, $D \subseteq X$ open, $x \in D$, $f: D \to Y$
For $v \in X \setminus \{0\}$ we define the directional derivative of f at x (in direction v) by $\frac{\partial f}{\partial v}(x) = \lim_{\substack{t \to 0 \\ t \in \mathbb{R}}} \frac{1}{t}(f(x+tv) - f(x))$ (as limit in Y)
if it exists. We call f Gateaux differentiable at x if $\frac{\partial f}{\partial v}(x)$ exists $\forall v \in X \setminus \{0\}$
If $X = \mathbb{R}^p$ and $Y = \mathbb{R}^m$, with $\|\cdot\|_2$ and $(e_i)_{i=1}^{p}$ the canonical basis of $\mathbb{R}^p$ we say that $\frac{\partial f}{\partial e_i}(x)$ is the "i'th partial derivative of f at x" (if it exists). If $\frac{\partial f}{\partial e_i}(x)$ exists $\forall i = 1, \dots, p$, we say that f is partially differentiable (we also use the notation $\frac{\partial f}{\partial x_i}(x)$ where $x_i = (x_1, \dots, x_p)^T$) ($f(x,y) \rightsquigarrow \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$)
If directional/partial derivative exists at every $x \in D$, f is called partially/directionally/Gateaux differentiable

If $f$ is partially differentiable at $x$

- For $Y = \mathbb{R}$, we define the "gradient" of $f$ at $x$ as
  $(\text{grad } f)(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \dots, \frac{\partial f}{\partial x_p}(x)\right)^T$
- For $Y = \mathbb{R}^m$, we define the Jacobian of $f$ at $x$

$$\frac{\partial f}{\partial x}(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_p}(x) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_p}(x) \end{bmatrix} \in \mathbb{R}^{m \times p}$$

6.8 => Facts:

- $X = Y = \mathbb{R}$, $D \subseteq \mathbb{R}$ open, $f: D \to \mathbb{R}$ is differentiable at $x$
  $\Leftrightarrow$ $f$ Gateaux differentiable $\Leftrightarrow$ $f$ is partially differentiable $(v = 1)$
- Since $D$, $v \in X / \{0\}$ is open $f_v: (-t_0, t_0) \to \mathbb{R}$, $t \mapsto f(x + tv)$ for some $t_0 > 0$ is well defined. (for $t \in (-t_0, t_0) \overset{t \text{ small}}{\Rightarrow} x + tv \in B_{t_0}(x) \Rightarrow x + tv \in D$
  $\Rightarrow \frac{\partial f}{\partial x}(x) = f_v'(0)$ if that exists)
- partial derivatives are computed by "freezing" all other variables except for $x_i$ and then differentiate
  $\frac{\partial f}{\partial x_i}(x)$ = derivative of $z \mapsto f(x_1, \dots, z, \dots, x_p)$ i'th component

6.9 => Examples:

a) $f: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x^2 + y^2$, Gateaux differentiable because
   $\forall v \in \mathbb{R}^2 \setminus \{0\}$, $f_v$ is a composistion of differential functions
   $\Rightarrow$ $f$ partially differentiable $\frac{\partial f}{\partial x}(x,y) = 2x$, $\frac{\partial f}{\partial y}(x,y) = 2y$

b) $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

   is Gateaux differentiable at $(x,y) \neq (0,0)$ as composistion of continuous functions $(x,y) = (0,0)$ needs more consideration
   $f$ partially differentiable, by $\frac{\partial f}{\partial x}(0,0) = \lim_{t \to 0} \frac{1}{t}(f(0+t,0) - f(0,0)) = 0 = \frac{\partial f}{\partial y}(0,0)$
   $f$ not Gateaux differentiable by $v = \binom{1}{1}$: $\frac{\partial f}{\partial v}(0,0) = \lim_{t \to 0} \frac{1}{t}(\underbrace{f(0+t, 0+t)}_{= \frac{t^2}{2t^2} = \frac{1}{2}} - 0) = \infty$
   (note that $f$ is not continuous at $(0,0)$ either)

c) $g(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

   $\Rightarrow g$ partially differentiable (as above)
   $v \in \mathbb{R} \setminus \{0\}$: $\frac{\partial f}{\partial v}(0,0) = \frac{v_1 v_2^2}{v_1^2 + v_2^2}$ $\Rightarrow$ $g$ Gateaux differentiable
   (note $g$ is continuous)

6.10 => **Definition: Continuous partial differentiable**

Let $X = \mathbb{R}^p$, $Y = \mathbb{R}^m$, $D \subseteq \mathbb{R}^p$ open, $x \in D$. A function $f: D \to \mathbb{R}^m$ is called continuously partially differentiable at $x$ if: $\{B_r = D\}$

1) $f$ is partially differentiable on a ball $B_r(x) \subseteq D$ for some $r > 0$
2) $B_r(x) \to \mathbb{R}^m$, $y \mapsto \frac{\partial f}{\partial x_i}(y)$ is continuous at $x_i$ for all $i = 1, \dots, p$ $\{D \to \mathbb{R}^m\}$

6.11 => Example:

a) $g(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$, $g: \mathbb{R}^2 \to \mathbb{R}$

is partially differentiable on $D$, $\frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial x}(0,0)$

$\frac{\partial f}{\partial x}(x,y) = \frac{y^2(x^2+y) - xy^2 2x}{(x^2+y^2)^2}$, $\frac{\partial f}{\partial y}(x,y) = \frac{2yx(x^2+y^2) - xy^2 2y}{(x^2+y^2)^2}$

It turns out that these are not continuous at $(0,0)$

b) $f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

turns out to be continuously partially differentiable (pp(3), ps(7))

6.12 => **Definition: (Total/Fréchet) differential**

Let $X, Y$ normed spaced (over $\mathbb{R}$), $D \subseteq X$ open, $x \in D$, $f: D \to Y$

Then $f$ is called (totally/Fréchet) differentiable at $x$ if

$\exists r > 0: B_r(x) \subseteq$, $\emptyset: B_r(x) \to \mathbb{R}$, $A \in \mathcal{L}(X,Y)$ such that

1) $\lim_{h \to 0} \frac{\emptyset(h)}{\|h\|_X} = 0$ error function
2) $f(x+h) = f(x) + A(h) + \emptyset(h)$ for all $h \in B_r(x)$ *

and the mapping $df(x) :=$ is called the differential of $f$ at $x$

=> Note: for $X = Y = \mathbb{R} \overset{*}{\iff} \underbrace{\frac{f(x+h) - f(x)}{h}}_{f'(x)} = A(1) + \frac{\emptyset h}{h}$

$h \to 0: \quad f'(x) = A$

6.13 => Facts:

- $A$ is uniquely defined if it exists
- If $f$ is differentiable at $x$, then $f$ is Gateaux differentiable at $x$ and $\frac{\partial f}{\partial v}(x) = A(v)$ $\forall v \in X \setminus \{0\}$

If $X = \mathbb{R}^p$, $Y = \mathbb{R}^m$: $A$ is represented by the Jacobian $\frac{\partial f}{\partial e_i}(x) = A(e_i)$ for $i = 1, \dots, p$

6.14 => **Theorem:** If $f$ is differentiable $\overset{\text{at } x}{\Rightarrow}$ $f$ continuous

6.13 => proposition: properties of $df$ (under the assumptions of 6.12)

Let $f$ be differentiable at $x \in D$

1) $f$ is Gateaux differentiable and $\forall v \in X\setminus\{0\}$ $\frac{\partial f}{\partial v}(x) = A(v) = df(x)(v)$

(also shows that $df$ is well defined)

2) If $X = \mathbb{R}^p$: $f$ is partially differentiable and $\frac{\partial f}{\partial x_i}(x) = df(x)(e_i)$ $\forall i = 1, \dots, p$

3) $f$ is continuous at $x$

4) If also $g: D \to Y$ differentiable at $x$, $\lambda \in \mathbb{R}$ then $\lambda f + g$ differentiable at $x$

↳ proof:

1) let $v \in X\setminus\{0\}$ and consider $f_v(t) \overset{def}{=} f(x+tv)$ and

$\frac{1}{t}(f_v(t) - f_v(0)) = \frac{1}{t}(f(x+tv) - f(x)) \overset{f\text{ diff.}}{=} \frac{1}{t}A(tv) + \phi(tv)$ for $tv \in B_r(0) \Leftrightarrow \|tv\| < r$

$= A(v) + \frac{\phi(tv)}{\|tv\|_X}\|v\|_X \xrightarrow{t \to 0} A(v) + 0 = A(v)$

3) Note: $f$ partially differentiable wouldn't imply $f$ continuous

We need to show $\lim_{y \to x} f(y) = f(x)$. If $(y-x) \in B_r(0) \Leftrightarrow \|y-x\|_X < r$ then

$f(y) - f(x) \overset{f\text{ diff}}{=} f(x) + A(y-x) + \phi(y-x) - f(x) = A(y-x) + \phi(y-x)$

$\|A(y-x) + \phi(y-x)\|_X \overset{\Delta}{\leq} \underbrace{\|A(y-x)\|_X}_{\overset{6.5}{\leq} c\|y-x\|_Y} + \underbrace{\|\phi(y-x)\|_Y}_{\xrightarrow{y \to x} 0 \text{ (def } \phi)} \xrightarrow{y \to x} 0$

=> $f$ continuous

6.14 => Facts:

- $f$ differentiable => $f$ continuous

  $f$ partially differentiable $\not\Rightarrow$ $f$ continuous
- Watch out definition Gateaux differentiable in literature
- The set $\mathcal{L}(X,Y) = \{A : X \to Y \text{ linear, bounded}\}$ becomes a vector space with $(\lambda A + B)(v) = \lambda A(v) + B(v)$ $\forall v \in X$ and $\|A\|_{\mathcal{L}(X,Y)} := \sup_{v \in X\setminus\{0\}} \frac{\|A(v)\|_Y}{\|v\|_X}$ is a norm
- Note: in contrast to $X = Y = \mathbb{R}$, the derivative $df(x)$ $(\in \mathcal{L}(X,Y))$ is a different object then $f(x) \in Y$

Also note: for $f$ differentiable $(\forall x \in D)$, then we can consider $x \mapsto df(x)$

$D \to \mathcal{L}(X,Y)$

6.15 => Definition: A differentiable function $f: D \to Y$ is called continuously differentiable if $x \mapsto df(x)$ is continuous from $D \subseteq X$ to $\mathcal{L}(X,Y)$

6.16 => Theorem: Let $X = \mathbb{R}^p$ $(Y = \mathbb{R}^m)$. A function $f: D \subseteq \mathbb{R}^p \to Y$, $D$ open has continuous partial derivatives $\Leftrightarrow$ $f$ is continuously differentiable

↳ proof:

"=>" First let $x \in D$ and show that $f$ differentiable at $x$.

By 6.13 the only candidate for $A (= df(x))$ is given by

$v \in \mathbb{R}^p$: $A(v) = \sum_{i=1}^{p} v_i \frac{\partial f}{\partial x_i}(x)$ where $v = \begin{pmatrix} v_1 \\ \vdots \\ v_p \end{pmatrix} \in \mathbb{R}^p = \sum_{i=1}^{p} v_i e_i$

=> $A \in \mathcal{L}(\mathbb{R}^p, Y)$ because $A$ linear and 6.6

To show $f(x+h)=f(x)+A(h)+\phi(h)$ $\forall h\in B_r(0)$ and some $r>0$ for some $\phi: B_r(0)\to Y$. Hence define $\phi(h)=f(x+h)-f(x)-A(h)$ where $r$ is chosen such that $B_r\subseteq D$ (possible since $D$ is open)

To show $\frac{\|\phi(h)\|_Y}{\|h\|_2}\to 0$ as $h\to 0$

Define $x_j=x+\sum_{i=1}^{j-1}h_ie_i$ where $h=\begin{pmatrix}h_1\\ \vdots\\ h_p\end{pmatrix}$ for $j=2,\dots,p$

$(x_2=x+h_1e_1),\ x_3=x+h_1e_1+h_2e_2\ \dots)$

$\Rightarrow x+h=x+h_1e_1+h_2e_2+\dots+h_pe_p=x_p+h_pe_p$

Idea: FTC for functions in one variable

$f(x+h)-f(x)-\underbrace{A(h)}=f(x+h)-f(x_p)+f(x_p)-f(x)-\sum_{i=1}^{p}h_i\frac{\partial f}{\partial x_i}(x)$

$=f(x+h)-f(x+h-h_pe_p)-h_p\frac{\partial f}{\partial x_p}(x)+f(x_p)-f(x)-\sum_{i=1}^{p-1}h_i\frac{\partial f}{\partial x_i}(x)$

$\overset{FTC}{=}\int_0^{h_p}\frac{\partial f}{\partial x_p}(x+h-se_p)ds-h_p\frac{\partial f}{\partial x_p}(x)+f(x_p)-f(x)-\sum_{i=1}^{p-1}h_i\frac{\partial f}{\partial x_i}(x)$

$=\int_0^{h_p}\frac{\partial f}{\partial x_p}(x_p+se_p)-\frac{\partial f}{\partial x_p}(x)\,ds+\dots R$

By induction: $f(x+h)-f(x)-A(h)=\sum_{i=1}^{p}\int_0^{h_i}\frac{\partial f}{\partial x_i}(x_i+se_i)-\frac{\partial f}{\partial x_i}(x)ds$

$\|f(x+h)-f(x)-A(h)\|\overset{\Delta}{\leq}\sum_{i=1}^{p}\int_0^{h_1}\underbrace{\|\frac{\partial f}{\partial x_i}(x_i+se_i)-\frac{\partial f}{\partial x_i}(x_i)\|_Y}_{\leq \max_{s\in[0,h_i]}\|\ \ "\ \ \|\text{ by continuity}}ds$

$\leq\sum_{i=1}^{p}h_i\max_{s\in[0,h_i]}\|\ \ "\ \ \|$

$\overset{C.S}{\leq}\|h\|_2\cdot\left(\sum_{i=1}^{p}\max_{s\in[0,h_i]}\|\ \ "\ \ \|^2\right)^{1/2}$

$\Rightarrow\frac{\|\phi(h)\|_Y}{\|h\|_2}\leq\left(\sum_{i=1}^{p}\max_{s\in[0,h_i]}\|\frac{\partial f}{\partial x_i}(x_i+se_i-\frac{\partial f}{\partial x_i}(x)\|_Y^2\right)^{1/2}$

$\forall i=1,\dots,p:\ \max_{s\in[0,h_i]}\|\ \ "\ \ \|\to 0$ as $h_i\to 0$ (because $\frac{\partial f}{\partial x_i}$ continuous)

Since finitely many $i$, the same is true for the sum if $h_i\to 0$ $\forall i=1,\dots,p \Leftrightarrow h\to 0$ in $(\mathbb{R}^p,\|.\|_2)$

6.17 $\Rightarrow$ **Theorem: Chain rule**

Let $X,Y,Z$ normed spaces, $D\subseteq X$, $G\subseteq Y$, open with $x\in D$, $f: D\to Y$, $g: G\subseteq Y\to Z$ such that $f(x)\in G$, $f(D)\in G$. If $f$ is differentiable at $x$ and $g$ differentiable at $f(x)$, then $g\circ f: D\to Z$ differentiable at $x$ and $d(g\circ f)(x)=(dg(f(x)))(df(x))=dg(f(x))\circ df(x)$

(If $x=y$ $(g\circ f)'(x)=g'(f(x))\cdot f'(x)$ )

↳ proof: Since $f$ and $g$ are differentiable $\exists r,r'>0$, $A\in\mathcal{L}(X,Y)$, $B\in\mathcal{L}(Y,Z)$ and such that $\phi_f(h)\overset{1}{=}f(x+h)-f(x)-A(h)$, $h\in B_r(0_X)$ and

$\phi_g(\tilde{h})\overset{2}{=}g(f(x+\tilde{h})-g(x)-B(\tilde{h})$, $\tilde{h}\in B_r(0_Y)$ yield

$\frac{\|\phi_f(h)\|_Y}{\|h\|_X}\to 0$, $\frac{\|\phi_g(\tilde{h})\|_Z}{\|\tilde{h}\|_Y}\to 0$ as $h,\tilde{h}\to 0$

$\phi_{g\circ f}(h)=g(f(x+h))-g(f(x))-(B\circ A)(h)$

$\overset{1}{=}g(f(x)+A(h)+\phi_f(h))-g(f(x))-(B\circ A)(h)$ where $\tilde{h}:=A(h)+\phi_f(h)$

$\overset{2}{=}g(f(x))+B(\tilde{h})+\phi_g(\tilde{h})-g(f(x))-(B\circ A)(h)$

$\Rightarrow\|B(\tilde{h})-(B\circ A)(h)\|_Z=\|B(\tilde{h}-A(h)\|_Z=\|B(\phi_f(h))\|_Z\leq\|B\|_{\mathcal{L}(Y,Z)}\|\phi_f(h)\|_Y$

$\Rightarrow\frac{\|B(\tilde{h})-(B\circ A)(h)\|_Z}{\|h\|_X}\to 0$ as $h\to 0_X$

$\frac{\|\phi_g(\tilde{h})\|_Z}{\|h\|_X}=\underbrace{\frac{\|\phi_g(\tilde{h})\|_Z}{\|\tilde{h}\|_Y}}_{\to 0}\cdot\frac{\|\tilde{h}\|_Y}{\|h\|_X}$ $\frac{\|\tilde{h}\|_Y}{\|h\|_X}=\frac{\|A(h)+\phi_f(h)\|_Y}{\|h\|_X}\leq\frac{\|A(h)\|_Y}{\|h\|_X}+\frac{\|\phi_f(h)\|_X}{\|h\|_X}\leq\|A\|+\varepsilon$

$\Rightarrow 0\cdot\|A\|+\varepsilon=0$

## 6.3 Higher order derivatives & Taylors theorem

6.18 => **Definition:** Let $D \subseteq X$ open, $X, Y$ normed spaces, $f: D \to Y$. We say that

- $f$ is twice/$k+1$ differentiable at $x \in D$ if $f^k$ differentiable on $D$ and $df: D \to \mathcal{L}(X,Y): y \mapsto d^k f$ is differentiable at $x$. We call $d(df)(x) \overset{def}{=} d^2 f$ the second derivative of $f$ at $x$.
- If $f$ is twice differentiable on $D$ and $d^2 f$ continuous, we say $f \in C^2$
- $f$ is twice partially differentiable at $x \in D$ if $f$ is partially differentiable on $D$ and all partial derivatives $y \mapsto \frac{\partial f}{\partial x_i}(y)$ are partially differentiable $(\forall i = 1, \dots, p)$ at $x$
- $f$ is called continuously twice partially differentiable if this holds for all $x \in D$ and $y \mapsto \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_i}(y)$ is continuous $(\forall i,j = 1, \dots, p)$

6.19 => Facts:

- The properties of the total derivative generalise to higher order derivatives. For instance, for $v_1, v_2 \in X \setminus \{0\}$, $f: D \subseteq X \to Y$ twice differentiable at $x \Rightarrow ((d^2 f)(x)(v_2))(v_1) = \left(\frac{\partial}{\partial v_1} \frac{\partial f}{\partial v_2}\right)(x)$
- We use the notation $((d^k f(x)(v_k)(v_{k-1}) \dots (v_1)) = (d^k f)(v_k \dots v_1)$ If $v_k = \dots = v_1$ $(d^k f)(x) v^k$

6.20 => Theorem: Schwarz

If $f$ is twice differentiable at $x$, then $(d^2 f)(x)$ is symmetric, that is $(d^2 f)(x)(v_1, v_2) = (d^2 f)(x)(v_2, v_1)$

↳ proof: For $f \in C^2$, $X = \mathbb{R}^p$. Consider $f(x + te_i + se_j) - f(x + te_i)$ for $t, s$ real $\overset{FTC}{=} \int_0^s \frac{\partial}{\partial e_j} f(x + te_i + re_j) dr$. Take partial derivative wrt $e_i$

$\Rightarrow \frac{\partial f}{\partial e_i}(x + te_i + se_j) - \frac{\partial f}{\partial e_i}(x + te_i) = \frac{\partial}{\partial e_i} \int_0^s \frac{\partial}{\partial e_j} f(x + te_i + re_j) dr$