(Class) Test-2: Analysis II

Statistics and Analysis (202001350)

25-October-2023, 13:45 – 15:15, CR-2M Total Points : 20

All answers must be motivated.
Approach to a solution is equally important as the final answer.
Use of an electronic calculator or a book is not allowed.
Good Luck!

1. We define X to be the space consisting of the open interval $(0,1)$ united with the point $\{\pi\}$, i.e., $X = (0,1) \cup \{\pi\}$. On X we define the standard metric of $\mathbb{R}$, i.e., $\rho(a,b) = |a-b|$.
   (a) Does there exists in (X,ρ) an open subset consisting of one point? If yes, give at least one of these sets, and a proof showing that it is open. If no, explain why. [2]
   (b) Is the subset $(0,1)$ a compact subset of (X,ρ)? [2]
   (c) On X we define the function
   $$f(x) = \begin{cases} x^2 & x \in (0,1), \\ x^2+1 & x \notin (0,1). \end{cases}$$
   Is f continuous from (X,ρ) to $\mathbb{R}$? [2]
2. Let (Y,τ) be a non-empty metric space.
   (a) Let $\{y_n, n \in \mathbb{N}\}$ be a Cauchy sequence in (Y,τ). Prove that it is a bounded sequence. [2]
   (b) Give the definition of (Y,τ) being connected. [2]
3. Let $f : \mathbb{R} \to \mathbb{R}^n$ be a C^1-function. Show that $g(t) := \|f(t)\|^2 f(2t)$ (the Euclidian squared norm of $f(t)$ times $f(2t)$) is differentiable from $\mathbb{R}$ to $\mathbb{R}^n$, and determine its derivative. [2]
4. Given the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined as $g(x,y) = x^3y^2 + 1$. Determine the tangent hyperplane at $(\mathbf{a}, g(\mathbf{a}))$, where $\mathbf{a} = (1,2)$. [2]
5. Given the function $f : \mathbb{R}^2 \to \mathbb{R}^2$:
   $$f(x,y) = \begin{bmatrix} xe^x\cos(y) - e^x\sin(y) \\ e^x\cos(y) + xe^x\sin(y) \end{bmatrix}.$$
   (a) Show that f is differentiable in $\mathbb{R}^2$, and determine its total derivative. [2]
   (b) Prove that there exists an f^{-1} which maps (e,e) to $(1,2\pi)$ and is differentiable in some nonempty open set containing (e,e). Compute the total derivative of this function in the point (e,e). [3]
   (c) Does the inverse function exist globally? [1]

Grade:	$\dfrac{\text{score on test}}{20} \times 9 + 1$	(rounded off to one decimal place)