

(Class) Test-1: Analysis II
202001350

28 September 2023, 13:45—15:15, Therm-1

Total Points : 19

All answers must be motivated.

Approach to a solution is equally important as the final answer.

Use of an electronic calculator or a book is not allowed.

Good Luck!

1. (a) Give the definition of convergence of a series of real numbers. [2]
(b) The two students JWP and HJZ are discussing the convergence of the series

$$\sum_{k=2}^{\infty} \log\left(\frac{k+1}{k}\right).$$

The student HJZ says that it diverges, whereas JWP claims it is converging. JWP has the following reasoning for it. Namely, you write

$$\log\left(\frac{k+1}{k}\right) = \log(k+1) - \log(k)$$

and now it has become a telescopic series, and thus converging.

Who of the students is correct, and why? [3]

2. Let $a_k \geq 0$ and let $\sum_{k=0}^{\infty} a_k$ be a convergent series. Prove that

$$\sum_{k=0}^{\infty} [e^{a_k} - 1]$$

is also convergent. [3]

3. On the interval $[-\pi, \pi]$, the function $f(x) = x^2 - \pi^2$ can be expressed in a Fourier cosine series:

$$f(x) = -\frac{2}{3}\pi^2 + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2} \cos(kx) \quad x \in [-\pi, \pi].$$

- (a) Does the following equality hold for all $y \in [0, \pi]$ [3]

$$\int_0^y f(x) dx = -\frac{2}{3}\pi^2 y + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^3} \sin(ky)?$$

- (b) Calculate the outcome of the following sum [2]

$$\sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell + 1)^3}.$$

4. (a) Find the convergence interval I of the power series: [3]

$$f(x) = \sum_{k=1}^{\infty} \frac{e^k}{k} x^k.$$

- (b) Show that $f(x)$ is analytic in any open interval within I . [2]

- (c) Determine the value of $f^{(10)}(0)$? [1]

Note that standard mathematical expressions, like $\sin(1)$, 234^5 , or $67890!$ are allowed to be used in your answer.

Grade: $\frac{\text{score on test}}{19} \times 9 + 1$ (rounded off to one decimal place)
