Analysis 1

Chapter 1: The real numbers

1.1 ⇒ **Definition**: Let M be a set. A relation R is called a partial ordering on M if:
1) R is reflexive ($xRx \ \forall x \in M$)
2) R is antisymmetric ($xRy \wedge yRx \Rightarrow x=y$)
3) R is transitive ($\forall x,y,z \in M: xRy \wedge yRz \Rightarrow xRz$

additionally if $\forall x,y \in M: xRy \vee yRx$ then (M,R) is linearly ordered (totally ordered)

1.2 ⇒ **Definition**: Let $(M, \oplus, \odot)$ be a field. We say that the field is ordered if there exists an M such that (M,R) is linearly ordered and:
1) $\forall x,y,z \in M: xRy \Rightarrow (x \oplus z) \, R \, (y \oplus z)$
2) $0_\oplus R x \wedge 0_\oplus R y \Rightarrow 0_\oplus R (x \odot y)$

1.3 ⇒ Example: $M = \mathbb{Q}$ with it's natrual field structure

"$\leq$" let $\frac{p}{q}, \frac{\tilde{p}}{\tilde{q}} \in \mathbb{Q}$ $\quad (q, \tilde{q} \in \mathbb{N}^*)$

define $\frac{p}{q} \leq \frac{\tilde{p}}{\tilde{q}} :\Leftrightarrow \overbrace{p \cdot \tilde{q}}^{\in \mathbb{Z}} \overset{?}{\leq} \overbrace{\tilde{p} \cdot q}^{\in \mathbb{Z}} :\Leftrightarrow \exists n \in \mathbb{N}: \tilde{p} \odot q = S^n(p \odot \tilde{q})$

↳ $\leq$ reflexive: $\forall x \in \mathbb{Q}: x \overset{?}{\leq} x$

$x \leq x \Leftrightarrow \tilde{p} \odot q \leq p \odot \tilde{q}$

$\overset{def}{\Leftrightarrow} \exists n \in \mathbb{N}: S^n(\tilde{p} \odot q) = (p \odot \tilde{q})$

(check well-definedness)

↳ the rest is to be checked

$\Rightarrow (\mathbb{Q}, \boxplus, \boxdot, \leq)$ is a ordered field

1.4 ⇒ **Definition**: The real numbers $\mathbb{R}$ are an ordered field which satisfies the completeness axiom, that is $\forall x,y$ a non-empty subset of $\mathbb{R}$ with $\forall x \in X, y \in Y: x \leq y$ there exists $c \in R: \forall x \in X, \forall y \in Y: x \leq c \leq y$

1.5 ⇒ **Remarks**:
1) Is this "$\mathbb{R}$ unique"

↳ For now: unclear! Later: yes, for some extent.

2) This is an axiomatic way of defining $\mathbb{R}$.
3) Consistency: at this moment it is not clear whether such $\mathbb{R}$ exists
4) Dedekind cuts

5) $\mathbb{Q}$ fails to satisfy the completeness axiom (next week)
6) Literature: many equivalent varients of the completeness axiom (supremum axiom)

Simple facts of $\mathbb{R}$

1.6 $\Rightarrow$ Facts:

1) $\exists!$ zero element with respect to "+"
   $\exists!$ one element with respect to "$\cdot$"
2) let A and B be real numbers ($\mathbb{R}$). Then the equation $a+x=b$ has unique solution $x=b+(-a)$
   If $a \neq 0_+$ then the equation $a \cdot x = b$ has unique solution $x = b \cdot a^{-1}$
3) $x \in \mathbb{R}: x \cdot 0_+ = 0_+ \cdot x = 0_+$
   ↳ proof: $x \cdot 0 = x(0+0) = x \cdot 0 + x \cdot 0$
   neutral element wrt '+' ; distribution
   $\Rightarrow \underbrace{x \cdot 0 + (-x \cdot 0)}_{=0} = x \cdot 0$
   $-x \cdot 0$ is the inverse element of $x \cdot 0$
4) $x \cdot y = 0_+ \Rightarrow (x=0) \vee (y=0)$
   ↳ proof: Suppose $y \neq 0$, apply 2
5) · $-x = (-1) \cdot x$
   · $(-1) \cdot (-x) = x$
   · $(-x) \cdot (-x) = x \cdot x$

1.7 $\Rightarrow$ Facts about order:

$\Rightarrow$ Notation: $\leq$ called less or equal
$\geq$ called greater or equal

$\Rightarrow$ Definition: $x, y \in \mathbb{R}$: $x < y :\overset{def}{\Leftrightarrow} (x \leq y) \wedge (x \neq y)$
↳ analogously $x > y$
"stricly less then/greater then"

1) $x, y \in \mathbb{R}$ we always have either $x<y$ or $x=y$ or $x>y$ (follows from linearly field and definition of "<" and ">")

2) (a) $(x<y \wedge y\leq z) \Rightarrow (x<z)$

(b) $(x\leq y \wedge y<z) \Rightarrow (x<z)$

↳ proof: $(x<y\wedge y\leq z) \Leftrightarrow ((x\leq y)\wedge(x\neq y)\wedge(y\leq z))$

$\Rightarrow ((x\leq y)\wedge(y\leq z))$

transivity ordering $\Rightarrow x\leq z$ antisymmetric ordering

Suppose $x=z$: $(x\leq y)\wedge(y<z) \Leftrightarrow (z\leq y)\wedge(y<z) \Leftrightarrow (z\leq y)\wedge(y\leq z)\wedge(y\neq z)$

$\Leftrightarrow (z=y)\wedge(y\neq z)$ ↯

3) • $x<y,\ z\in\mathbb{R} \Rightarrow x+z<y+z$

• $0<x \Rightarrow -x<0$

• $(x\leq y)\wedge(z\leq w) \Rightarrow x+z\leq y+w$

4) $0<x \wedge 0<y \Rightarrow 0<x\cdot y$

* $x<0 \wedge y<0 \Rightarrow 0<x\cdot y$

$x<0 \wedge y>0 \Rightarrow x\cdot y<0$

$x<y \wedge 0<z \Rightarrow xz<yz$

$x<y \wedge z<0 \Rightarrow yz<xz$

5) $0<1$

↳ proof: $1\in\mathbb{R}\setminus\{0\} \Leftrightarrow 1\neq 0$

if $1<0 \overset{*}{\Rightarrow} ((1<0)\wedge(1<0)) \Rightarrow 0<1\cdot 1$

$\Rightarrow 0<1$ ↯

So $\Rightarrow 0<1$

6) $0<x \Rightarrow 0<x^{-1}$

$(0<x)\wedge(x<y) \Rightarrow 0<y^{-1} \wedge y^{-1}<x^{-1}$

Completeness axiom & consequences

1.8 ⇒ **Definition**: A subset X of $\mathbb{R}$ is called bounded above if $\exists c\in\mathbb{R}: \forall x\in X: x\leq c$ and bounded below if $\exists c\in\mathbb{R}: \forall x\in X: x\geq c$

Any such C is called an upper/lower bound of X

↳ An upper/lower bound of X is called maximal/minimal if it is an element of X

↳ maximal and minimal element may not exist

1.9 => Definition: (sup/inf) let x be a subset of $\mathbb{R}$. The smallest upperbound of x is called supremum of x denoted by sup x. Similarly, the greatest lower bound of x is called infimum denoted by inf x.

↳ x is bounded if bounded above and below

↳ If x is not bounded of above/below we set up

$\sup x = \infty$

$\inf x = -\infty$

=> Theorem: Any bounded above subset of $\mathbb{R}$ has a Supremum (supreme axiom)

Completeness axiom:

$\forall X, Y$ non-empty subsets of $\mathbb{R}$ with:

$\forall x \in X, \forall y \in Y : x \leq y \Rightarrow \exists c \in \mathbb{R} : \forall x \in X \wedge \forall y \in Y : x \leq c \leq y \, (x \leq c \wedge c \leq y)$

=> Definition: (sup/inf) let M be a subset of $\mathbb{R}$, then the smallest upperbound of M is called the supremum of M, the greatest lower bound of M is called the infemum

↳ Note: $\sup M = \min\{c \in \mathbb{R} \mid \underbrace{c \text{ is upper bound of } M}_{\forall x \in M : x \leq c}\}$

$\inf M = \max\{c \in \mathbb{R} \mid \forall x \in M, \ c \leq x\}$

↳ minimal/maximal elements, if they exists, are unique:

Let x,y be minimal in $M \Rightarrow (\forall z \in M : x \leq z \wedge y \leq z) \wedge x \in M \wedge y \in M$

$(z = x, z = y) \Rightarrow x \leq y \wedge y \leq x$

$(\leq \text{ antisym.}) \Rightarrow x = y$

↳ If min/max exist, they equal sup/inf

If x minimal in $M \Rightarrow x = \inf M$

1.10 => Theorem: any non-empty bounded $\left\{\begin{matrix}\text{above}\\ \text{below}\end{matrix}\right.$ set has a unique $\left\{\begin{matrix}\text{supremum}\\ \text{infemum}\end{matrix}\right.$

↳ proof: Since minimal/maximal elements are unique it is enough to show their existence.

Consider a non-empty bounded set X and
$Y: \{c \in \mathbb{R} : \forall x \in X : x \leq c\}$ (Y is non-empty because X is bounded above)
let $x \in X, y \in Y$: $x \leq y$ because by definition of Y, y is an upper bound of x $(c=y)$

$\overset{\text{complete axiom}}{\Rightarrow} \exists c \in \mathbb{R} : \forall x \in X, \forall y \in Y : x \overset{1}{\leq} c \overset{2}{\leq} y$

claim: This c is the sup of X
By '1' c is an upper bound of X
and by '2' any upperbound y of x satisfies $c \leq y$
$\Rightarrow C = \underset{\uparrow \text{Facts}}{\min Y} = \operatorname{Sup} X$

↳ proof inf is analogously

1.11 ⇒ Remark: The implication of the theorem implies the completeness-axiom (see Hw/Ex)

1.12 ⇒ Example $M \{x \in \mathbb{R} : 0 \leq x < 1\}$
M is bounded (0 lower bound, 1 upper bound)
$0 = \min M$
$0 \in M, \forall x \in M\ 0 \leq x$
$\Rightarrow 0 = \inf M$
claim: $\sup M = 1$: It's clear that $\sup M \leq 1$ (Facts)
↳ We need to show that $^{*}\forall x \in M, \exists q \in M : x < q < 1$
(this means $\forall x < 1$: x is not an upper bound)
How to show $*$? Construct q
Claim: $q = 2^{-1} \cdot (x+1)$ satisfies $*$ $\qquad (2 := \overset{\mathbb{R}}{1} + \overset{\mathbb{R}}{1})$
proof to show $x < q \wedge q < 1$

$$\Updownarrow$$

$x < 2^{-1} \cdot (x+1)$
$\overset{\text{def.}}{\Leftrightarrow} x < (1+1) \cdot (x+1)$
$\overset{\text{order facts}}{\Leftrightarrow} (1+1)x < x+1$
$\overset{\mathbb{R}\text{ field}}{\Leftrightarrow} \underbrace{1 \cdot x}_{x} + \underbrace{1 \cdot x}_{x} < x+1$
$\overset{+(-x)\text{ order}}{\Leftrightarrow} x < 1$, this is true since $x \in M$
$q < 1$: $1 = \sup M$, but 1 is not maximal since $1 \notin M$

1.13 $\Rightarrow$ Example existence of square root

↳ Claim: $\forall y \in \mathbb{R}\ (y \geq 0) \Rightarrow (\exists x \in \mathbb{R} : x \cdot x = y \wedge x > 0)$

Moreover such x is unique for y

Such x is denoted b $\sqrt{y}$ or $y^{\frac{1}{2}}$

↳ Strategy: Define a set M in $\mathbb{R}$, such that $x = \sup M$

↳ Candidate: $M = \{z \in \mathbb{R} ,\ 0 \leq z^2 \leq y\}$ "=" $[0, \sqrt{y}]$

↳ Claim: $x := \sup M$ satisfies $\overset{x^2}{x \cdot x} = y$

As M is bounded and non-empty $\Rightarrow x$ exists

We can assume that $0 \leq y < 1$

$\overset{\text{Facts order}}{\Rightarrow} 0 \leq z^2 \leq z < 1$ for all $z \in M$

By ordering facts: $x^2 < y \,\dot\vee\, x^2 = y \,\dot\vee\, x^2 > y$

↳ Case 1: $x^2 < y$: (we show that then $x \neq \sup M$)

Let $\varepsilon := \min\{(2x+1)^{-1}(y - x^2)\}$ since $x^2 < y \Rightarrow \varepsilon > 0$

$\Rightarrow z := x + \varepsilon > x$ show that $z \in M \Rightarrow$ contradiction to $x = \sup M$

$z \in M \overset{\text{Facts order}}{\Rightarrow} 0 \leq z^2 \leq y \Leftrightarrow z^2 \leq y$

$z^2 = (x+\varepsilon)(x+\varepsilon) = (x+\varepsilon)x + (x+\varepsilon)\varepsilon$

$= x^2 + (1+1)\varepsilon x + \varepsilon^2$

$= x^2 + (2x+\varepsilon)\varepsilon \underset{\text{Facts order}}{\leq} x^2 + (2x+1)\varepsilon \leq x^2 + y - x^2$

$= y$

↳ Case 2: $x^2 > y$

$\Rightarrow$ Uniqueness: Let $x, \tilde{x} \in \mathbb{R}$. $x \geq 0 \wedge \tilde{x} \geq 0 \wedge x = y, \tilde{x} = y$

$\Rightarrow x^2 = \tilde{x}^2$

$(\tilde{x} \geq x \wedge x \geq x^2 \Rightarrow x = \tilde{x})$

1.14 $\Rightarrow$ Lemma: $\forall y \in \mathbb{R}, n \in \mathbb{N}\ (y \geq 0) \Rightarrow \exists x \in \mathbb{R} : (x \geq 0,\ \underbrace{x \cdot x \cdot x \cdot x \cdots x}_{n\text{-times}} = y)$

↳ Proof: similar to example 'existence of square root' !

↳ $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ in $\mathbb{R}$

$\Rightarrow$ Recall Tom: we constructed $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$

inclusion: $\mathrm{rat}(\mathrm{int}(\mathbb{N})) \subseteq \mathbb{Q}$

$\mathrm{int}\ \mathbb{N} \to \mathbb{Z} : n \mapsto (n, 0)$

ets

Define $f: \mathbb{Z} \to \mathbb{R} : [(m,n)] \mapsto \begin{cases} s^{m-n-1}(1_{\mathbb{R}}) & \text{if } m>n \\ -s^{n-m-1}(1_{\mathbb{R}}) & \text{if } n>m \\ 0_{\mathbb{R}} & \text{if } m=n \end{cases}$ where

$S: \mathbb{R} \to \mathbb{R}$, $x \mapsto x+1_{\mathbb{R}}$ $\quad \left(S^0 : \mathbb{R} \to \mathbb{R}, \; x \mapsto x \right)$

↳ e.g. $[1,0] \mapsto S^{1-0-1}(1_{\mathbb{R}}) = 1_{\mathbb{R}}$

$$g : \underset{\mathbb{Z}\times\mathbb{Z}^*}{\overset{}{\underset{\cap\!|}{\mathbb{Q}}}} \to \mathbb{R} : [(p,q)] \mapsto f(p) \cdot f(q)^{-1}$$

"g sends $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ to the conomical elements in $\mathbb{R}$"

1.15 ⇒ Lemma: The function $g: \mathbb{Q} \mapsto \mathbb{R}$ is well-defined and
$\forall x,y \in \mathbb{Q} \begin{cases} g(x \boxplus y) = g(x) + g(y) \\ g(x \boxdot y) = g(x) \cdot g(y) \end{cases}$
and g is injective
↳ proof: ∞

1.16 ⇒ **Definition:**
- $g(\mathbb{N}^*)$ is called the natural numbers in $\mathbb{R}$
  $\{x \in \mathbb{R} : \exists n \in \mathbb{N}^* : g(n) = x\}$
  ↳ we write $\mathbb{N}^*$ for $g(\mathbb{N}^*)$
- $g(\mathbb{N})$ is called the natural numbers with 0 in $\mathbb{R}$
  ↳ we write $\mathbb{N}$ for $g(\mathbb{N})$
- $g(\mathbb{Z})$ is called the integers in $\mathbb{R}$
  ↳ we write $\mathbb{Z}$ for $g(\mathbb{Z})$
- $g(\mathbb{Q})$ is called the rationals in $\mathbb{R}$
  ↳ we write $\mathbb{Q}$ for $g(\mathbb{Q})$

1.17 ⇒ **Definition:** (Zorich's definition of $\mathbb{N}$) A subset of $\mathbb{R}$ with the property that $\forall x \in M : x+1_{\mathbb{R}} \in M$ is called inductive

1.18 ⇒ Lemma: $g(\mathbb{N})$ is the smallest inductive set in $\mathbb{R}$ containing $0_{\mathbb{R}}$
$g(\mathbb{N}^*)$ is the smallest inductive set in $\mathbb{R}$ containing $1_{\mathbb{R}}$

$\forall$ inductive set M containing $0_{\mathbb{R}}/1_{\mathbb{R}} \Rightarrow g(\mathbb{N}) \subseteq M$
$\Leftrightarrow g(\mathbb{N})$ is the intersection of all inductive sets containing $0_{\mathbb{R}}/1_{\mathbb{R}}$

↳ proof: $g(\mathbb{N})$ is inductive and contains $0_{\mathbb{R}}$ (by def. of g)
Let M be inductive and containing $0_{\mathbb{R}}$
Consider $S = \{n \in \mathbb{N} : g(n) \in M\}$
S contains 0 ($\emptyset$ in $\mathbb{N}$)
Furthermore $\forall n \in S$ the succesor also lies in S (by def. of g)
$\overset{\text{by def. } \mathbb{N}}{\Rightarrow} S = \mathbb{N}$
$\overset{\text{by def. } S}{\Rightarrow} g(\mathbb{N}) \subseteq M$ $\square$

"Zorich defines $\mathbb{N}^*$ as the smallest inductive set containing $1_{\mathbb{R}}$"

1.19 ⇒ Remarks: · Since g is injective: no ambiguity in our notation $g(\mathbb{N}^*) = \mathbb{N}^*$, $g(\mathbb{Z}) = \mathbb{Z}$, $g(\mathbb{Q}) = \mathbb{Q}$
· In analysis the natural numbers are $\mathbb{N}^*$ (excluding $0_{\mathbb{R}}$)
· Zorich's approach is in line with ours by Lemma above

1.20 ⇒ Corollary: Principle of math induction
Any subset of $\mathbb{N}^*$, which is inductive and contains $1_{\mathbb{R}}$ equals $\mathbb{N}^*$ ($\forall E \subseteq \mathbb{N}^* : (1_{\mathbb{R}} \in E \wedge E \text{ inductive}) \Rightarrow E = \mathbb{N}^*$

1.21 ↳ Example "The sum of the first n natural numbers equals $n \cdot (n + 1_{\mathbb{R}}) \cdot 2^{-1} = \frac{n(n+1)}{2}$" (Gauß)
Let's define $\tilde{S}(n) = n \cdot (n+1) \cdot 2^{-1}$ $\tilde{S} : \mathbb{N}^* \to \mathbb{R}$ and
$S(n) = n \cdot (n+1) \cdot 2^{-1}$ $S : \mathbb{N}^* \to \mathbb{R}$ sum of the first n natural numbers.
We need to show $S = \tilde{S}$. To do so, Consider
$M = \{n \in \mathbb{N}^* : S(n) = \tilde{S}(n)$
We want to show that $1 \in M \wedge M \text{ inductive} \overset{\text{coro}}{\Rightarrow} M = \mathbb{N}^*$
$1 \in M :\Leftrightarrow S(1) = \tilde{S}(1) = 1 \cdot (1+1) \cdot 2^{-1} = 1$ ✓ (induction beginning)
M inductive : if $\underline{n \in M} \overset{!}{\Rightarrow} \underline{n+1 \in M}$ $\forall n \in M$
$\Leftrightarrow S(n) = \tilde{S}(n)$ $\Leftrightarrow S(n+1) = \tilde{S}(n+1)$

$$
\begin{aligned}
S(n+1) &\overset{\text{def. } S}{=} S(n) + n + 1 \\
&\overset{\text{IB}}{=} \tilde{S}(n) + n + 1 \\
&= n(n+1) \cdot 2^{-1} + (n+1) = \tilde{S}(n+1)
\end{aligned}
$$

1.22 => **Definition: The irrational numbers**

Every element in $\mathbb{R}$, which is not in $\mathbb{Q}$ ($g(\mathbb{Q})$) is called irrational (irrational numbers = $\mathbb{R} \setminus \mathbb{Q}$)

1.23 => Theorem: irrational numbers

The squareroot of 2 is irrational

↳ proof: by "existence of the square root" the square root of 2 exist uniquely as positive unit

$(\forall y \in \mathbb{R} : y > 0 \Rightarrow \exists! \, x \in \mathbb{R} : (x \geq 0 \land x \cdot x = y))$

We need to show that $x = \sqrt{2} = 2^{\frac{1}{2}}$ is not in $\mathbb{Q}$

Suppose $x = p \cdot q^{-1}$ for $p \in \mathbb{Z}, q \in \mathbb{Z}^*$

without loss of generality (wlog) we assume that $p, q \geq 0$

Moreover assume that exactly one of p and q is even and the other odd

(if both were odd, by $(p \cdot q^{-1})^2 = 2 \Rightarrow p^2 = 2q^2 \Rightarrow p^2$ even $\Rightarrow p$ even)

$\Rightarrow p^2 = 2q^2$: case 1 p even: p^2 is divisible by 4, while $2q^2$ is not

Case 2 q even: p^2 must be even → contradiction □

1.24 => Facts:

1) $\forall n, m \in \mathbb{N}^* : \underline{n+m} \in \mathbb{N}^*$ (by prop. of p, lemm. 1.15)

$g(n) + g(m)$

$f(n) + f(m) = f(n+m)$ (note difference to book)

2) $\forall n \in \mathbb{N}^* : n \neq 1 \Rightarrow n - 1 \in \mathbb{N}^*$

3) $\forall n \in \mathbb{N}^* : \min\{x \in \mathbb{N}^* : n < x\} = n+1$

4) $\forall n, m \in \mathbb{N}^* : n < m \Rightarrow n+1 \leq m$

5) $\forall n \in \mathbb{N}^* \, \nexists x \in \mathbb{N}^* : n < x < n+1$

6) $n-1$ is the immediate predeccesor f n for every $n \in \mathbb{N}^*, n \neq 1$

7) Every subset of $\mathbb{N}$ has a minimal element

5. The Archimedeon Principle

1.25 => Facts:

1) Every non-empty bounded above subset M of $\mathbb{N}^*$ has a maximal element

↳ proof: by sup-'axiom' $\exists s = \sup M$

$\overset{\text{def sup}}{\Rightarrow} \exists n \in M^* : s - 1 < n \leq s \Rightarrow n \leq s < n+1 \overset{\text{Facts } \mathbb{N}^*}{\Rightarrow} s = n \Rightarrow \operatorname{Max} M = \sup M = s$

2) $\mathbb{N}^*$ is not bounded above

↳ proof: Suppose $c \in \mathbb{R}$ was an upperbound, $\forall n \in \mathbb{N}^*: n < c$

$\mathbb{N}^*$ is inductive $\Rightarrow n+1 \in \mathbb{N}^*$

Ussing (1) $n+1 > c$, which contradicts that c is an upper bound

3) $\mathbb{Z}$ is neither bounded below or above

4) Any non-empty bounded below subset subset of $\mathbb{Z}$ (or $\mathbb{N}$) has a minimal element

1.26 $\Rightarrow$ proposition: Principle of Archimedes

$\forall x \in \mathbb{R}, \forall h > 0, \exists! k \in \mathbb{Z}: (k-1)h \leq x < kh$

↳ proof: consider $M = \{n \in \mathbb{Z} : x < nh\}$

non-empty because bounded below $(n \in \mathbb{Z} : \frac{x}{h} < n)$

$\overset{\text{Fact 4}}{\Rightarrow} \exists! k = \min(M)$

$k = \min(M) \overset{\text{def } M}{\Leftrightarrow} k \in M \wedge (k-1) \notin M$

$\Updownarrow$ $x < kh \Leftrightarrow (k-1)h \leq x$

Since minima are unique, such k is unique

1.27 $\Rightarrow$ Facts

1) $\forall \varepsilon > 0 \; \exists n \in \mathbb{N}^*: 0 < \frac{1}{n} < \varepsilon$

↳ proof: $0 < \frac{1}{n} < \varepsilon \Leftrightarrow 0 < \underbrace{1}_{=:x} < \underbrace{n \cdot \varepsilon}_{=:h}$

apply archimedes $\Rightarrow \exists! n \in \mathbb{Z}: 1 < n \cdot \varepsilon$

Since $\varepsilon > 0 \wedge 0 < 1 \Rightarrow n \in \mathbb{N}^*$

2) $\forall x \in \mathbb{R}: (x \geq 0 \wedge \forall n \in \mathbb{N}^* \; x < \frac{1}{n}) \Rightarrow x = 0$ $[\forall n \in \mathbb{N}^*: 0 \leq x < \frac{1}{n} \Rightarrow x = 0]$

↳ proof: Suppose $x > 0$. By first point $\exists n \in \mathbb{N}^*: \frac{1}{n} < x$ ↯

3) $\forall a, b \in \mathbb{R}: (a < b) \Rightarrow \exists r \in \mathbb{Q}: a < r < b$

↳ proof: Let $a, b \in \mathbb{R}$ such that $a < b$. By the first fact, $\exists n \in \mathbb{N}^*: \frac{1}{n} < b - a$

By the Archimedian principle (with $h = \frac{1}{n}$) there exists $k \in \mathbb{Z}$ s.t.

$\frac{k-1}{n} \leq a \leq \frac{k}{n}$

$\overset{\text{above}}{\Rightarrow} a < \frac{m}{n} < b$

4) $\forall x \in \mathbb{R}\ \exists! z \in \mathbb{Z} : z \leq x < z+1$
"We call this z the integer part of x"
"$[\cdot] : x \to \mathbb{Z}$ floor function"

6. The completeness axiom & consequences II

1.28 => **Definition**: A sequence, let X be a set. A function $A : \mathbb{N}^* \to \mathcal{P}(x)$
such a function is called a sequence of sets
↳ We use the short-hand notation $\underbrace{A_1, A_2, \ldots}_{\text{subsets of } x} \hat{=} A$

1.29 => **Definition**: A sequence of sets $A_1, A_2, \ldots$ is called nested if
$A : \mathbb{N}^* \to \mathcal{P}(x)$ is degreasing with respect to the natural inclusion ordering of $\mathcal{P}(x)$. That is $\forall n \in \mathbb{N}^* : A_n \supseteq A_{n+1}$
↳ short-hand notation: $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$

=> Theorem: Nested intervals principle
For every nested sequence I_k of closed intervals in $\mathbb{R}$ there exists $c \in \mathbb{R}$ which lies in every interval of the sequence
Given $I_1, I_2, \ldots$ nested closed intervals => $\bigcap_{k \in \mathbb{N}^*} I_k$ is not empty
If, additionally $\forall \varepsilon > 0\ \exists k \in \mathbb{N}^* :$ lenght of $I_k < \varepsilon$ $(|I_k| < \varepsilon)$.
Then such c is unique
↳ proof: plan: use completeness axiom (directly)
Define the sets $A = \{a_k : k \in \mathbb{N}^*\}$, $B = \{b_k : k \in \mathbb{N}^*\}$
(Say $I_k = [a_k, b_k]$ for some $a_k, b_k \in \mathbb{R} : a_k \leq b_k$)
=> because we have closed intervals A, B are non-empty
=> because we have nested intervals $\forall a \in A, b \in B : a \leq b$
$\overset{\text{complete axiom}}{\Longrightarrow}$ $\exists c \in \mathbb{R} : \forall a \in A, \forall b \in B : a \leq c \leq b$
=> $\forall k \in \mathbb{N}^* : a_k \leq c \leq b_k$
<=> $c \in I_k\ \forall k \in \mathbb{N}^*$
<=> $c \in \bigcap_{k \in \mathbb{N}^*} I_k$
<=> $\bigcap_k I_k \neq \emptyset$

Under the additional assumption, suppose that c_1 and c_2 are different elements in the intersection of all I_k's.

Let $\varepsilon = |c_1 - c_2| > 0$, then $\exists k \in \mathbb{N}^*$ such that $|I_k| < \frac{\varepsilon}{2}$
$\Rightarrow |I_k| = \max I_k - \min I_k \geq |c_1 - c_2| = \varepsilon$ ↯

7. Countable vs. uncountable

1.30 ⇒ **Definition**: Let M be a set. We say that:
- M is finite if there exist a bounded above subset N of $\mathbb{N}^*$ and a bijection $f: N \to M$
- M is countable, if there exist a bijection $f: \mathbb{N}^* \to M$
- M is at most countable, if M is either finite or bijective

↳ Clearly $\mathbb{N}^*$ is countable (as f can be chosen to be the identity mapping), $\mathbb{Z}$ is also countable ($f: \mathbb{N}^* \to \mathbb{Z}$), $\mathbb{Q}$ turns out to be countable too (counter).

1.31 ⇒ proposition: Any infinite subset of a countable set is countable

1) The finite or countable union of finite or countable sets is at most countable

↳ example: $\mathbb{Z} = \mathbb{N}^* \cup \{0\} \cup \{-k : k \in \mathbb{N}^*\}$

↳ proof Zorich 75

1.32 ⇒ Theorem: The real numbers are not countable

↳ proof: Suppose $\mathbb{R}$ was countable

⇒ a sequence $f_1, f_2, \ldots$ in $\mathbb{R}$ such that $f: \mathbb{N}^* \to \mathbb{R}$ is bijective

Goal: find element in $\mathbb{R}$ which is not in the image of the sequence.

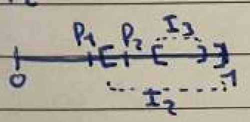
It's enough to show that $[0,1]$ is uncountable

Assume $f_1, f_2, \ldots \in [0,1]$

Consider f_1 and choose an closed interval not containing f_1, I_2

In I_2 choose a subinterval not containing f_2

Call this $I_3 \ldots \overset{1.29}{\Rightarrow} \exists c \in \bigcap_{k \in \mathbb{N}^*} I_k \wedge c \notin f(\mathbb{N}^*)$



Chapter 2: Sequences

2.1 => **Definition**: Let X be a set. A function $x: \mathbb{N}^* \to X$ is called a sequence in X. We use the notation $(x_n)_{n\in\mathbb{N}^*}$ (or short-hand (x_n))
↳ The set of sequences in X is denoted by $X^{\mathbb{N}^*}$
↳ In the case $X = \mathbb{R}$, we also say "x is a sequence of real numbers" (Zorich: "numerical sequences")

2.2 => **Definition**: A general distance function
Let X be a set. Then a function $d: X \times X \to [0,\infty)$ is called a distance function or metric on X if:
i) $\forall x,y \in X: d(x,y) = 0 \iff x = y$
ii) $\forall x,y \in X: \ d(x,y) = d(y,x)$ (symmetric)
iii) $\forall x,y,z \in X: d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality)

The pair (X,d) is called a metric space

2.3 => Examples:
a) most important: $X = \mathbb{R}, d(x,y) := |x-y|$
↳ proof: !
b) the simplest: "discrete metric"
$$\text{let } X \neq \emptyset,\ d(x,y) = d_0(x,y) = \begin{cases} 1 \text{ if } x \neq y & \forall x,y \in X \\ 0 \text{ if } x = y & \forall x,y \in X \end{cases}$$
↳ proof: !
c) abstract example:
Let (X,d) be a metric space and $Y \subset X$ and
$$d|_{Y\times Y} : Y \times Y \to [0,\infty)$$
$$(x,y) \mapsto d(x,y)$$
=> $(Y, d|_{X\times X})$ metric space
↳ proof: !
d) most important (general): $X = \mathbb{R}^d = \overbrace{\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}}^{d\text{-times}}$
$$d_2(x,y) = \left(\sum_{j=1}^{d} |x_j - y_j|^2\right)^{1/2}$$
=> $(\mathbb{R}^d, d_2)$ metric space
$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}, y = \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix}$$
↳ proof: !

$\Rightarrow$ Recall: for $z \in \mathbb{R}$: $|z| = \begin{cases} z & \text{if } z \geq 0 \\ -z & \text{if } z < 0 \end{cases}$

$\forall a,b \in \mathbb{R}: |a+b| \leq |a| + |b|$

$|a| = 0 \Leftrightarrow a = 0$

$|a-b| = |b-a| \Leftarrow |a| = |-a|$

$\Rightarrow$ Note: the differences of general sets X (belonging to (X,d) metric) and special cases like $X = \mathbb{R}$ $d = d_2$ (from example) this has consequences for the structure of $X^{\mathbb{N}^*}$

↳ e.g: Sequences in $\mathbb{R}$ can be added or multplied

$x, y \in \mathbb{R}^{\mathbb{N}^*}$ define $x+y \in \mathbb{R}^{\mathbb{N}^*}$

$x \cdot y \in \mathbb{R}^{\mathbb{N}^*}$

by $(x+y)(n) \overset{def}{=} x(n) + y(n)$

$(x \cdot y)(n) \overset{def}{=} x(n) \cdot y(n)$

2.4 $\Rightarrow$ Example:

$X = \mathbb{R}$, $d(x,y) = |x-y|$

$x_n = n^2$, $n \in \mathbb{N}^*$ $\quad (x: \mathbb{N}^* \to \mathbb{R} \quad n \mapsto n^2)$

$\to (n^2)_{n \in \mathbb{N}^*}$

•) $x_n = \frac{1}{n}$ $n \in \mathbb{N}^*$

•) $(X,d) = (\mathbb{R}, d_0)$

$x_n = n^2 \to (n^2)_{n \in \mathbb{N}^*}$

$x_n = \frac{1}{n} \to (\frac{1}{n})_{n \in \mathbb{N}^*}$

2.5 $\Rightarrow$ Definitions (convergent, Cauchy, bounded sequences)

Let (X,d) be a metric space.

Then a sequence $(x_n)_{n \in \mathbb{N}^*}$ in X is called:

1) Convergent if:

$\exists x \in X \ \forall \varepsilon > 0 \ \exists N_\varepsilon \in \mathbb{N}^* \ \forall n \in \mathbb{N}^* \ (n \geq N \Rightarrow d(x_n, x) < \varepsilon)$

2) a cauchy sequence if:

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}^* \ \forall n,m \in \mathbb{N}^* \ (n,m \geq N \Rightarrow d(x_n, x_m) < \varepsilon)$

3) bounded if:

$\exists c \in X, M > 0 : \forall n \in \mathbb{N}^* \ d(c, x_n) < M$

2.6 $\Rightarrow$ Remark: (what does convergent mean?)

The sequence $(\frac{1}{n})_{n\in\mathbb{N}^*}$ converges in $\mathbb{R}$

$x=0 \Rightarrow \forall \varepsilon>0\ \exists N_\varepsilon \in \mathbb{N}^*\ \forall n \geq N_\varepsilon:$

$d(\frac{1}{n},0)=|\frac{1}{n}-0|<\varepsilon \Leftrightarrow \frac{1}{n}<\varepsilon \Leftrightarrow n>\frac{1}{\varepsilon}$

Choose: $N_\varepsilon=\min\{n\in\mathbb{N}^*: n>\frac{1}{\varepsilon}\}$ exist by fact:.... (or floor function)

Addition to definition 2.5:

If a sequence (x_n) converges, then "the x" is called the limit.

we then also say/write "(x_n) converges to x", $x_n \xrightarrow{n\to\infty} x$

("x_n tends to x as an goes to infinity")

If $(X,d)=(\mathbb{R},d_2)$, then we make no further references $(d_2(x,y)=|x-y|)$

2.7 $\Rightarrow$ proposition: let (X,d) be a metric space. Let $(x_n)_{n\in\mathbb{N}^*}$ be a Sequence in X

Then (x_n) convergent $\Rightarrow (x_n)$ cauchy sequence $\Rightarrow (x_n)$ bounded

Furthermore: • "Limits are unique", that means

$(x_n \xrightarrow{n\to\infty} x \wedge x_n \xrightarrow{n\to\infty} y) \Rightarrow x=y$

• $x_n \xrightarrow{n\to\infty} x$ in $(X,d) \Leftrightarrow (\overbrace{d(x_n,x)}^{\in\mathbb{R}})_{n\in\mathbb{N}^*} \xrightarrow{n\to\infty} 0$ in $(\mathbb{R},d_2)$

↳ proof: let $x_n \xrightarrow{n\to\infty} x$ in (X,d), we want to show that $(x_n)_{n\in\mathbb{N}^*}$ is cauchy

Consider $d(x_n,x_m) \underset{\triangle\text{-inequal.}}{\leq} d(x_n,x)+d(x,x_m)$

Since $x_n \xrightarrow{n\to\infty} x$ by assumption $\forall \frac{\varepsilon}{2}>0\ \exists N_\varepsilon \in \mathbb{N}^*: \forall n \in \mathbb{N}^*$

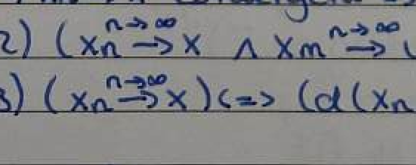
$n\geq N \Rightarrow d(x_n,x)<\frac{\varepsilon}{2}$

$\Rightarrow n\geq N \wedge m\geq N \Rightarrow d(x_n,x)<\frac{\varepsilon}{2} \wedge d(x,x_m)<\frac{\varepsilon}{2}$

$\Rightarrow \forall n,m\in\mathbb{N}^*: n\geq N \wedge m\geq N \Rightarrow d(x_n,x)+d(x,x_m)<\frac{\varepsilon}{2}$

$\Rightarrow d(x_n,x_m)<\varepsilon$

C.S. $\Rightarrow$ hold hw



$\varepsilon=\frac{d(x,y)}{2}$

(2.7) $\Rightarrow$ Recall: (X,d) metric space $(x_n)_{n\in\mathbb{N}^*}$ Sequence in X

1) $x_n \xrightarrow{n\to\infty} x$ convergent $\Rightarrow (x_n)$ cauchy $\overset{?\ \text{zorich}}{\Rightarrow} (x_n)$ bounded

2) $(x_n \xrightarrow{n\to\infty} x \wedge x_m \xrightarrow{n\to\infty} y) \Rightarrow x=y$

3) $(x_n \xrightarrow{n\to\infty} x) \Leftrightarrow (d(x_n,x))_{n\in\mathbb{N}^*}$ converge to 0 in $\mathbb{R}$ $(\mathbb{R},d_2)$

$x_n \xrightarrow{n\to\infty} x \overset{\text{def}}{\Leftrightarrow} \forall \varepsilon>0\ \exists N\in\mathbb{N}^*\ \forall n\geq N: d(x_n,x)<\varepsilon$

$d=d_2 \quad d_2(x,y)=|x-y| \quad x,y\in\mathbb{R}$

↳ proof (3): $z_n = d(x_n, x)$. consider $(z_n)_{n\in\mathbb{N}^*}$

$z_n \xrightarrow{n\to\infty} 0$ in $(\mathbb{R}, d_2)$

$\overset{\text{def.}}{\iff} \forall \varepsilon>0 \ \exists N \in \mathbb{N}^* \ \forall n \geq N \ \overbrace{|z_n - 0|}^{= d_2(z_n, 0)} < \varepsilon$

$|z_n - 0| \overset{\text{def } z_n}{=} d(x_n, x)$

$\overset{\text{def. conv.}}{\iff} x_n \xrightarrow{n\to\infty} x$ in (X, d)

↳ proof (2): use (3) to show (2)

let x, y be limits of $(x_n)_{n\in\mathbb{N}^*}$

$x = y \iff 0 = d(x,y) \overset{\triangle}{\leq} d(x, x_n) + d(x_n, y)$

by 3) $d(x,x_n) \to 0$, $d(x_n,y) \to 0$ as $n \to \infty$

$\Rightarrow d(x,y) = 0 \Rightarrow x = y$

⇒ Example: $X = \mathbb{R}$, $d = d_2$ $x_n = \frac{1}{n}$, $n \in \mathbb{N}^*$

(convergent to 0 because $\forall \varepsilon > 0 \ \exists N: N > \frac{1}{\varepsilon}$

(Archimedean principle)

2.8 ⇒ Facts on sequences in $\mathbb{R}$ ($(\mathbb{R}, d_2)$)

1. let $(x_n), (y_n)$ be sequences in $\mathbb{R}$, $x, y \in \mathbb{R}$

a) $x_n \xrightarrow{n\to\infty} x \wedge y_n \xrightarrow{n\to\infty} y \Rightarrow (x_n + y_n)_{n\in\mathbb{N}^*}$ converges to $x+y$

↳ proof: $x_n \xrightarrow{n\to\infty} x \iff \forall \varepsilon>0 \ \exists N \in \mathbb{N}^* \ \forall n \geq N: |x_n - x| < \varepsilon$

$y_n \xrightarrow{n\to\infty} y \iff \forall \varepsilon>0 \ \exists \tilde{N} \in \mathbb{N}^* \ \forall n \geq \tilde{N}: |y_n - y| < \varepsilon$

$\Rightarrow \forall \varepsilon > 0 \ \exists N' = \max\{N, \tilde{N}\} \ \forall n \geq N'$:

$|(x_n + y_n) - (x+y)| \leq |x_n - x| + |y_n - y| < 2\varepsilon$

b) $x_n \xrightarrow{n\to\infty} x \wedge y_n \xrightarrow{n\to\infty} y \Rightarrow (x_n \cdot y_n)_{n\in\mathbb{N}^*}$ converges to $x \cdot y$

↳ proof: one addition step needed

Assume $\forall \varepsilon>0 \ \exists N, \tilde{N} \in \mathbb{N}^*: \forall n \geq N \ |x_n - x| < \varepsilon$

$n \geq \tilde{N} \ |y_n - y| < \varepsilon$

$|x_n \cdot y_n - x \cdot y| + xy_n - xy_n = |\underbrace{x_n y_n - x y_n}_{= (x_n - x) y_n} + \underbrace{x y_n - x \cdot y}_{= x(y_n - y)}|$

$\overset{\triangle}{\leq} \underbrace{|y_n|}_{\leq \sup_n |y_n| = M} \underbrace{|x_n - x|}_{< \varepsilon (n \geq N)} + \underbrace{|x| \, |y_n - y|}_{\leq |x|\varepsilon (\text{if } n \geq \tilde{N})} = ⊛$

want: $\forall \tilde{\varepsilon} > 0 \ \exists N' \in \mathbb{N}^* \ \forall n \geq N' \ |x_n y_n - xy| < \varepsilon$

How to choose N' (depending on $\tilde{\varepsilon}$)?

(want $⊛ < \tilde{\varepsilon}$) this is achieved by

$⊛ < M\varepsilon + |x|\varepsilon$ for $n \geq \max\{N_\varepsilon, \tilde{N}_\varepsilon\}$

take instead of $N_\varepsilon, \tilde{N}_\varepsilon$: $N_{\tilde{\varepsilon}/2M}$, $\tilde{N}_{\tilde{\varepsilon}/2|x|}$ if $x \neq 0$

$N_{\tilde{\varepsilon}}$ if $x = 0$

c) if $y_n \neq 0 \ \forall n \in \mathbb{N}^*$

$x_n \xrightarrow{n\to\infty} x \wedge y_n \xrightarrow{n\to\infty} y \Rightarrow (\frac{x_n}{y_n})_{n\in\mathbb{N}^*}$ converges to $\frac{x}{y}$

↳ proof: check that $y_n \xrightarrow{n\to\infty} y \wedge \forall n \in \mathbb{N}^*: y_n \neq 0 \Rightarrow (\frac{1}{y_n} \xrightarrow{n\to\infty} \frac{1}{y})$

rest is clear from b) !

2. Let $x_n \xrightarrow{n\to\infty} x$, $y_n \xrightarrow{n\to\infty} y$ in $\mathbb{R}$

If $\exists N \in \mathbb{N}^*: \forall n \geq N \ x_n < y_n$, then $x \leq y$

(the same for $x_n \leq y_n \to x \leq y$) (nevers remains strict)

(Ex: $x = 0 \ \forall n \in \mathbb{N}^*$, $y_n = \frac{1}{n}$, $\forall n \in \mathbb{N}^*$)

↳ proof: zorich

3. Sandwich sqeeze theorem

If $x_n \overset{xxx}{\leq} y_n \overset{xxxx}{\leq} z_n$ for every $n \in \mathbb{N}^*$ (for some index N all $n \geq N$)

and $x_n \xrightarrow{n\to\infty} x$, $z_n \xrightarrow{n\to\infty} x$ then $y_n \xrightarrow{n\to\infty} x$

(ex: $x_n = 0 \ \forall n$, $z_n = \frac{1}{n} \ \forall n$, $y_n = \frac{1}{n^2} \ \forall n$

↳ proof: Suppose that $x_n \xrightarrow{n\to\infty} x$, $z_n \xrightarrow{n\to\infty} x$ and let $\varepsilon > 0$

$\Rightarrow \exists N' \in \mathbb{N}^*: x - \varepsilon < \overset{x}{x_n}$ for $n \geq N'$

$z_n < x + \varepsilon^{(xx)}$

$-\varepsilon \overset{!}{<} y_n - x \overset{!}{<} \varepsilon$

$x - \varepsilon \leq y_n \leq x + \varepsilon$ for all $n \geq N'$

$\uparrow$ x, xxx $\quad \uparrow$ x, xxxx

$|x_n - x| < \varepsilon \quad n \geq N'$

$\Leftrightarrow -\varepsilon < x_n - x < \varepsilon$

$\Leftrightarrow x - \varepsilon < x_n < x + \varepsilon$

4. Let $x_n \xrightarrow{n\to\infty} x$, $y_n \xrightarrow{n\to\infty} y$. Suppose $\exists N \in \mathbb{N}^*$ such that:

a) $x_n > y_n \Rightarrow x \geq y \quad \forall n \geq N$

b) $x_n > y_n \Rightarrow x \geq y \quad \forall n \geq N$

c) $x_n > y \Rightarrow x \geq y \quad \forall n \geq N$

d) $x_n \geq y \Rightarrow x \geq y \quad \forall n \geq N$

$\Rightarrow$ Difference between (x_n) convergent and (x_n) cauchy set

(x_n) is a cauchy set (in $\mathbb{R}$) $\overset{def}{\Leftrightarrow} \forall \varepsilon > 0 \ \exists N \in \mathbb{N}^* \ \forall n, m \geq N: |x_n - x_m| < \varepsilon$

(x_n) convergent $\overset{def}{\Leftrightarrow} \exists x \in \mathbb{R} \ \forall \varepsilon > 0 \ \exists N \in \mathbb{N}^* \ \forall n \geq N \ |x_n - x| < \varepsilon$

2. Characterisations of convergence

2.9 => Theorem: Cauchy criteria

Any cauchy sequence in $\mathbb{R}$ converges

$\overset{2.6}{\Longleftrightarrow}$ A sequence in $(\mathbb{R}, d_2)$ converges if and only if it is a cauchy set

↳ proof: by prop 2.6 (con => cauch) we only need to show that:

(x_n) cauchy $\Rightarrow$ (x_n) converges

Challenge is to find the limit

By prop 2.6: (x_n) cauchy $\Rightarrow (x_n)$ bounded explicitly

Given $\varepsilon > 0$ $\exists N \in \mathbb{N}^*: \forall n, m \geq N: |x_n - x_m| < \varepsilon$

$(m = N) \Leftrightarrow x_N - \varepsilon < x_n < x_N + \varepsilon \quad \forall n \geq N$

This shows that (x_n) is bounded

Define $a_n = \inf_{k \geq n} x_k \overset{\text{def}}{=} \inf\{x_k : k \geq n\}$

$$b_n = \sup_{k \geq n} x_k \qquad \text{for all } n \in \mathbb{N}^*$$

note by construction: $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$

$I_n := [a_n, b_n]$, $n \in \mathbb{N}^*$

$\Rightarrow I_n$ are nested intervals

$\overset{1.29}{\Longrightarrow} \exists x \in \mathbb{R}: x \in I_n \ \forall n \in \mathbb{N}^*$

To show: $x_n \xrightarrow{n \to \infty} x$

Since $a_n = \inf_{k \geq n} x_k \leq x \leq b_n = \sup_{k \geq n} x_k$ (by def. inf & sup)

$\Rightarrow |x - x_n| \leq b_n - a_n$

Moreover $|x - x_k| \leq b_n - a_n \quad \forall k \geq n$

$x_N - \varepsilon \underset{?}{\leq} \inf_{k \geq n} x_k \overset{\text{def}}{=} a_n \leq b_n \overset{\text{def}}{=} \sup x_k \underset{?}{\leq} x_n + \varepsilon \quad (n \geq N)$

$\Rightarrow b_n - a_n < 2\varepsilon$

$\Rightarrow \forall \varepsilon > 0 \ \exists N \in \mathbb{N}^*: \forall n \geq N: |x - x_n| < 2\varepsilon$

2.10 => Example:

"The statement of theorem 2.9 is not true if $X = \mathbb{Q}$, $d = d_2(\mathbb{Q}, d_2|_{\mathbb{Q}\times\mathbb{Q}})$

By Archimedean principle (Facts):

$\forall$ irrational number x (e.g. $\sqrt{2}$) $\exists$ seq. (x_n) in $\mathbb{Q}$: $x_n \xrightarrow{n \to \infty} x$

$x \notin \mathbb{Q}$, it does not converge in $(\mathbb{Q}, d_2|_{\mathbb{Q}\times\mathbb{Q}})$

=> Example:

Let $x_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ $n \in \mathbb{N}^*$, $(x_n)_{n\in\mathbb{N}^*}$ in $\mathbb{R} (= (\mathbb{R}, d_2)$ $d_2(x,y) = |x-y|)$

↳ Does this sequence converge?

By theorem 2.9 (con => cauchy), it's enough to show that (x_n) is not a cauchy set.

To see that (x_n) is not cauchy. Consider

$|x_{2n} - x| = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}$

$\geq n \cdot \frac{1}{2n} = \frac{1}{2}$

$\Rightarrow (x_n)$ is not a cauchy sequence $(\varepsilon = \frac{1}{2})$

3. Monotonic sequences & Euler's number

2.11 => **Definitions**: A sequence $(x_n)_{n\in\mathbb{N}^*}$ in $\mathbb{R}$ is called

1a) non-decreasing if $\forall n \in \mathbb{N}^*$ $x_n \leq x_{n+1}$

b) non-increasing if $\forall n \in \mathbb{N}^*$ $x_n \geq x_{n+1}$

2a) increasing if $\forall n \in \mathbb{N}^*$ $x_n < x_{n+1}$

b) decreasing if $\forall n \in \mathbb{N}^*$ $x_n > x_{n+1}$

3a) bounded above if $\{x_n : n \in \mathbb{N}^*\}$ is bounded above

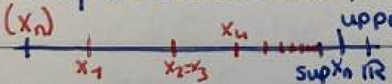
bounded below if $\{x_n : n \in \mathbb{N}^*\}$ is bounded below

4) monotonic if (x_n) is either (non)-increasing or (non)-decreasing

2.12 => Theorem: Weierstraß/monotonic convergence

Any bounded above, non-decreasing seq. in $\mathbb{R}$ converges to $\sup\{x_n : n \in \mathbb{N}^*\}$

Any bounded below, non-increasing seq. in $\mathbb{R}$ converges to $\inf\{x_n : n \in \mathbb{N}^*\}$



↳ proof: Let (x_n) is non-decreasing and bounded above.

Let S be the supremum (exists by ass. of boundedness)

By def. of sup: $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}^* : S - \varepsilon \leq x_N$

Since (x_n) is non-decreasing: $\forall n \geq N : x_N \leq x_n$

Since S is an upperbound: $\forall n \geq N : x_n < S + \varepsilon$

$\Bigg\}$ $\forall n \geq N$: $S - \varepsilon < x_n < S + \varepsilon$ $\Leftrightarrow \forall n \geq N : |S - x_n| < \varepsilon$

2.13 ⇒ Example: (Euler's number)

Let $y_n = (1+\frac{1}{n})^{n+1}$, $n \in \mathbb{N}^*$

Then $(y_n)_{n\in\mathbb{N}^*}$ is decreasing in $\mathbb{R}$

Also (y_n) is bounded below, because by using Bernoulli's inequality: $\forall x > -1, n \in \mathbb{N}^* : (1+x)^n \geq 1 + nx$

By $x := \frac{1}{n}$: $y_n \geq 1 + (n+1)\frac{1}{n} = 1 + \frac{n+1}{n} \geq 2 \quad \forall n \in \mathbb{N}^*$

$\overset{2.12}{\Rightarrow} \lim_{n\to\infty} y_n$ exists

Define $\lim_{n\to\infty}(1+\frac{1}{n})^n =: e$

$\underbrace{(1+\frac{1}{n})^n}_{= x_n} \underbrace{(1+\frac{1}{n})}_{\xrightarrow{n\to\infty} 1} = y_n$

2.14 ⇒ Examples:

a) Let $q > 1$. The sequence $(\frac{n}{q^n})$ converges to 0.

↳ proof: Theorem 2.12 can not be applied directly

$x_n = \frac{n}{q^n}$, $x_{n+1} = \frac{n+1}{q^{n+1}} = \frac{n+1}{nq} x_n$, $n \in \mathbb{N}^*$

If $\frac{n+1}{nq} < 1$, then (x_n) is decreasing

$\Leftrightarrow 1 + \frac{1}{n} < q$. Therefore $\exists N \in \mathbb{N}^* : \frac{n+1}{nq} < 1 \;\forall n \geq N$

Define $y_n = x_{n+N}$, $n \in \mathbb{N}^*$

⇒ $(y_n)_{n\in\mathbb{N}^*}$ is decreasing and bounded below as $y_n > 0 \;\forall n \in \mathbb{N}^*$

$\overset{2.12}{\Rightarrow} \lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n$ exists $=: x$

$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}(\underbrace{\frac{n+1}{nq}}_{z_n} x_n) = \frac{1}{q}\cdot x$, $\quad \lim_{n\to\infty} z_n = \frac{1}{q}$

$\Rightarrow (1 - \frac{1}{q})\cdot x = 0 \overset{q \neq 1}{\Rightarrow} x = 0$

b) $\sqrt[n]{n} \xrightarrow{n\to\infty} 1$ (see 1.14) This follows from $\frac{n}{q^n} \to 0$ $(q>1)$

c) $\forall a > 0 : \sqrt[n]{a} \xrightarrow{n\to\infty} 1$

4. Subsequences & lim inf / lim sup

2.15 ⇒ **Definition**: Let (x_n) be a sequence in the metric space (X, d) and $k : \mathbb{N}^* \to \mathbb{N}^*$ increasing, that is $k(n) < k(n+1) \;\forall n \in \mathbb{N}^*$ ("k is an increasing sequence in $\mathbb{N}^*$")

The function $x \circ k$ is called a subsequence of x

$n \mapsto x(k(n)) = x_{k(n)} = x_{k_n}$

=> Example: $x_n = (-1)^n,\ n\in\mathbb{N}^*$ $\quad (-1,1,-1,1,-1,\dots)$

$h(n) = 2n,\ n\in\mathbb{N}^* \Rightarrow x\circ h = (1,1,1,1,1,\dots)$

$h(n) = 2n+1,\ n\in\mathbb{N}^* \Rightarrow x\circ h = (-1,-1,-1,-1,-1,\dots)$

2.16 => **Definition**: A sequence in $\mathbb{R}$ is said to converge to:

$$\begin{cases} +\infty & \text{if for } \forall c\in\mathbb{R}\ \exists N\in\mathbb{N}^*: \forall n\geq N\ x_n > c \\ -\infty & \text{if for } \forall c\in\mathbb{R}\ \exists N\in\mathbb{N}^*: \forall n\leq N\ x_n < c \end{cases}$$

↳ Tutorial: (x_n) converges to $+\infty \Leftrightarrow$ every subsequence is not bounded above

2.17 => Lemma: Every sequence in $\mathbb{R}$ either has a convergent subsequence or a subsequence converging to $+\infty$ or $-\infty$

↳ proof: Tutorial, hints in notebook

2.18 => Facts:

a) Any bounded sequence has a convergent subsequence (Bolzano-Weierstraß)

↳ proof: follows from 2.17. By showing that no subsequence converging to $+/-\infty$ exists

b) (x_n) convergent in $(X,d) \Leftrightarrow (x_{h(n)})_{n\in\mathbb{N}^*}$ converges a subsequence $x_{h(n)}$ and limits are the same

c) For a sequence to converge "it doesn't matter what finitely sequence components do"

This means: Given $(x_n)_{n\in\mathbb{N}^*}$ in $\mathbb{R}$ and define

$$y_n \begin{cases} x_n & \text{for } n \text{ in a sequence. } S\subseteq\mathbb{N}^* \text{ with } \mathbb{N}^*\setminus S \text{ finite} \\ c_n & \text{else} \end{cases}$$

where $c_n, n\in\mathbb{N}^*\setminus S$ is arbitrary. Then (x_n) converges $\Leftrightarrow$ $y(n)$ converges and limits (if exists) are the same.

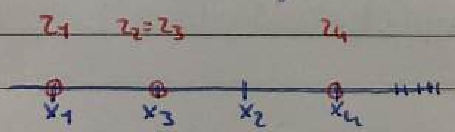
d) Given a sequence $(x_n)_{n\in\mathbb{N}^*}$ in $\mathbb{R}$.

define $z_n := \inf\{x_k : k\geq n\} \overset{def}{=} \inf\limits_{k\geq n} x_k$ (recall proof of 2.9)

Then $(z_n)_{n\in\mathbb{N}^*}$ is non-decreasing.

Similarly $\tilde{z}_n := \sup\limits_{k\geq n} x_k$ defines a non-increasing sequence $(\tilde{z}_n)_{n\in\mathbb{N}^*}$

$$\Rightarrow \begin{cases} (z_n) \text{ converges (to it's sup) or } (z_n) \text{ to } +\infty \\ (\tilde{z}_n) \text{ converges (to it's inf) or } (\tilde{z}_n) \text{ to } -\infty \end{cases}$$



2.19 ⇒ **Definition**: Given $(x_n)_{n\in\mathbb{N}^*}$ in $\mathbb{R}$, we call

$\lim_{n\to\infty} \inf_{k\geq n} x_k = \underline{\lim}_{n\to\infty} x_n$ the limit inferior of (x_n) and

$\lim_{n\to\infty} \sup_{k\geq n} x_k = \overline{\lim}_{n\to\infty} x_n$ the limit superior of (x_n)

2.20 ⇒ Examples:

a) $x_n = \frac{1}{n}, n\in\mathbb{N}^*$,

$\underline{\lim}_{n\to\infty} x_n = 0 = \overline{\lim}_{n\to\infty} x_n$

$\Updownarrow \qquad \Updownarrow$

$\lim_{n\to\infty} \underbrace{\inf_{k\geq n} x_n}_{=0} \qquad \lim_{n\to\infty} \underbrace{\sup_{k\geq n} x_n}_{=x_n} = 0$

by the monotone converge theorem and fact finitely many components doesn't matter

b) $x_n = (-1)^n \; n\in\mathbb{N}^*$,

$\underline{\lim}_{n\to\infty} x_n = -1 \qquad (\underbrace{-1,1,-1}_{\text{remove}},\underbrace{1,-1,1,-1,\dots}_{\inf=-1,\ \sup=1})$

$\overline{\lim}_{n\to\infty} x_n = +1$

c) $x_n = n, n\in\mathbb{N}^*$,

$\overline{\lim}_{n\to\infty} x_n = +\infty$

$\underline{\lim}_{n\to\infty} x_n = -\infty$

$\overset{\text{def}}{=} \lim_{n\to\infty} \underbrace{\inf_{k\geq n} k}_{=n} = +\infty$

2.21 ⇒ **Definition**: Given a sequence $(x_n)_{n\in\mathbb{N}^*}$ in some metric space (X,d), we call the limit of a subsequence of $(x_n)_{n\in\mathbb{N}^*}$ a partial limit (accumulation point) of $(x_n)_{n\in\mathbb{N}^*}$

2.22 ⇒ Proposistion: The limit $\left\{\begin{matrix}\text{inferior}\\ \text{superior}\end{matrix}\right\}$ of a sequence in $\mathbb{R}$ is the $\left\{\begin{matrix}\text{smallest}\\ \text{largest}\end{matrix}\right\}$ partial limit of the sequence

↳ proof: We prove the statement for $\underline{\lim}_{n\to\infty} x_n$ for a given sequence $(x_n)_{n\in\mathbb{N}^*}$

Define $\underline{\lim}_{n\to\infty} x_n = \lim_{n\to\infty} \underbrace{\inf_{k\geq n} x_k}$

(z_n) non-decreasing

$i = \sup_{n\in\mathbb{N}^*} z_n \overset{\text{monotone conv.}}{=} \lim_{n\to\infty} z_n$

Suppose $i\in\mathbb{R}$ (sequence (x_n) bounded below)

How to find a subsequence with limit i:

Choose indices (k_n) inductively such that:

- $k_n < k_{n+1}$ (definition z_n and definition of inf)
- $z_{k_n} \leq x_{k_n} < z_{k_n} + \frac{1}{n}$

$$\Rightarrow \lim_{n\to\infty} z_{k_n} = \lim_{n\to\infty} \left(z_{k_n} + \frac{1}{n}\right)$$

$$\overset{\text{"}}{i} \overset{\text{Sqeeze}}{\Longrightarrow} \lim_{n\to\infty} x_{k_n} = i$$

($\overline{\lim_{n\to\infty}}$ in the same way)

So we have found a subsequence $(x_{k_n})_{n\in\mathbb{N}^*}$ which converges to the $\underline{\lim}_{n\to\infty} x_n$

Why is this the smallest:

$\forall \varepsilon > 0\ \exists N \in \mathbb{N}^*\ \forall n \geq N\ |i - x_{k_n}| < \varepsilon$

know: $z_n \xrightarrow{n\to\infty} i$

$\Rightarrow \forall \varepsilon > 0\ \exists n \in \mathbb{N}^* : i - \varepsilon < z_n \overset{\text{def}}{=} \inf_{k\geq n} x_k \leq x_k \quad \forall k \geq n$

$\Rightarrow i - \varepsilon < x_k \quad \forall k \geq n$

$\Rightarrow$ no subsequence of $(x_n)_{n\in\mathbb{N}^*}$ can converge to a limit strictly less then i

5. Series (infinite sums)

7.23 $\Rightarrow$ **Definition**: let $(x_n)_{n\in\mathbb{N}^*}$ be a sequence in $\mathbb{R}$.
Then the sequence defined by

$S_n = x_1 + x_2 + \ldots + x_n$

$\overset{\text{def}}{=} \sum_{k=1}^{n} x_k,\ n \in \mathbb{N}^*$

is called the sequence of partial sums of the series $\sum_{k=1}^{\infty} x_k$

We say that the series $\sum_{k=1}^{\infty} x_k$ converges if $(s_n)_{n\in\mathbb{N}^*}$ converges.

Let $(a_j)_{j\in\mathbb{N}^*}$ be a sequence in $\mathbb{R}$. The sequence $(s_n)_{n\in\mathbb{N}^*}$
$S_n := \sum_{j=1}^{n} a_j,\ n \in \mathbb{N}^*$ is called a series (of real numbers).

↳ We write $\sum_{j=1}^{\infty} a_j$ for both the series as well as its limits (if they exists)

↳ If $\sum_{j=1}^{\infty}$ does not converges, we say it diverges. This includes $\sum_{j=1}^{\infty} a_j$ converges to infinity ($+/- \infty$)

↳ The a_j's are called "terms of the series"

2.24 $\Rightarrow$ Facts:

1) If $\sum_{j=1}^{\infty} a_j$ converges, then $a_j \xrightarrow{j\to\infty} 0$ (divergence test)
($a_j \not\to 0 \Rightarrow \sum_{j=1}^{\infty} a_j$ diverges)

2) Let $(a_k)_k, (\tilde{a}_k)_k$ be sequences in $\mathbb{R}$ such that $\{k \in \mathbb{N}^* : a_k \neq \tilde{a}_k\}$ is finite. Then $\sum_{j=1}^{\infty} a_j$ converges if and only if $\sum_{j=1}^{\infty} \tilde{a}_j$ converges. However in that case limit's need not be the same.

↳ proof: Define $k := \max\{k \in \mathbb{N}^* : a_k \neq \tilde{a}_k\} \in \mathbb{N}^*$

$\forall n \geq k: \sum_{j=1}^{n} a_j = \sum_{j=1}^{k} a_j + \sum_{j=k+1}^{n} \underbrace{a_j}_{=\tilde{a}_j \text{ since } j>k}$

$= \sum_{j=1}^{k} (a_j - \tilde{a}_j) + \sum_{j=1}^{n} \tilde{a}_j$ (Fact 2.8)

3) A series $\sum_{j=1}^{\infty} a_j$ of real numbers converges if and only if:
$\forall \varepsilon > 0\ \exists N \in \mathbb{N}^*\ \forall n, m \geq N\ (m \geq n \Rightarrow |\sum_{j=n}^{m} a_j| < \varepsilon)$

↳ proof: (cauchy seq. $\Leftrightarrow$ conv. seq in $\mathbb{R}$) statement follows from $(S_n)_{n\in\mathbb{N}^*}$ is a cauchy sequence.

"Cauchy criteria for series"

$S_m - S_{n-1} \overset{\text{def. partial sums}}{=} \sum_{j=1}^{m} a_j - \sum_{j=1}^{n-1} a_j$, $m \geq n \geq 2$
$= \sum_{j=n}^{m} a_j$

4) Let $(a_j)_{j\in\mathbb{N}^*}$ be a sequence of non-negative numbers ($a_j \geq 0$, $j \in \mathbb{N}^*$). Then $\sum_{j=1}^{\infty} a_j$ converges if and only if $(\underbrace{\sum_{j=1}^{n} a_j}_{=S_n})_{n\in\mathbb{N}^*}$ is bounded above.

↳ proof: This follows from monotone convergence (Weierstraß) theorem, because $(\sum_{j=1}^{n} a_j)_{n\in\mathbb{N}^*}$ is non-decreasing ($S_{n+1} = S_n + a_{n+1} \geq S_n$ $\forall n \in \mathbb{N}^*$)

5) Comparison lemma/test:
Let $(a_j)_j$ and $(b_j)_j$ be real sequences, such that $0 \leq a_j \leq b_j$ for all but finitely many $j \in \mathbb{N}^*$.

i) If $\sum_{j=1}^{\infty} b_j$ converges, then $\sum_{j=1}^{\infty} a_j$ converges

↳ proof: By fact (2) we can assume that $\forall j \in \mathbb{N}^*$ $0 \leq a_j \leq b_j$
if $\sum_{j=1}^{\infty} b_j$ convergent $\overset{(4)}{\Rightarrow} (\sum_{j=1}^{n} b_j)_{n\in\mathbb{N}^*}$ bounded above $\Rightarrow$
$(\sum_{j=1}^{n} a_j)_{n\in\mathbb{N}^*}$ bounded above $\overset{(4)}{\Rightarrow} \sum_{j=1}^{\infty} a_j$ converges

ii) If $\sum_{j=1}^{\infty} a_j$ diverges, then $\sum_{j=1}^{\infty} b_j$ diverges (write: $\sum_{j=1}^{\infty} b_j = \infty$)
↳ proof analogously by fact 4.

2.26 => Examples:

1) Let $q \setminus \{0\} \in \mathbb{R}$ by the formula $\sum_{j=0}^{n} q^j = 1+q+q^2+\dots+q^n$
$$= \begin{cases} \frac{1-q^{n+1}}{1-q} & \text{if } q \neq 1 \\ n+1 & \text{if } q = 1 \end{cases}$$
$\sum_{j=1}^{\infty} q^j$ converges $\iff |q|<1$ (see tutorial, $\sum_{j=m}^{\infty} q^j = \frac{q^m}{1-q}$, $m \in \mathbb{N}^*$)

2) $\sum_{j=1}^{\infty} \frac{1}{j}$ diverges because of fact (3), by noting that
$S_{2n} - S_{n-1} = \sum_{j=n}^{2n} \frac{1}{j} \geq n \cdot \frac{1}{2}$ $n \in \mathbb{N}^*$ (see 2.10) (Harmonic series)

3) $\sum_{j=1}^{\infty} \frac{1}{j^2}$ converges, because $\forall j \in \mathbb{N}^*, j>1$ $\frac{1}{j^2} < \frac{1}{j-1} \cdot \frac{1}{j}$
and since $\sum_{j=2}^{n} \frac{1}{(j-1)j} = \sum_{j=2}^{n} \left(\frac{1}{j-1} - \frac{1}{j}\right) = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right)$
$= 1 - \frac{1}{n} \to \infty \ (n \to \infty)$
By facts 2 and 5 $\sum_{j=1}^{\infty} \frac{1}{j^2}$ converges (with limit ≤ 1)

2.26 => Definition: A series $\sum_{j=1}^{\infty} a_j$ of real (or complex) numbers is called absolutely convergent if $\sum_{j=1}^{\infty} |a_j|$ converges

2.27 => proposistion: If $\sum_{j=1}^{\infty} a_j$ converges absolutely, it converges
↳ proof: $\overset{\text{conv. abs.}}{\Rightarrow}$ $\forall \varepsilon>0$ $\forall m,n \geq N$ $m \geq n \Rightarrow \sum_{j=n}^{m} |a_j| < \varepsilon$
$\overset{\text{triangle ineq.}}{\Rightarrow}$ $\forall \varepsilon>0$ $\forall m,n \geq N$ $m \geq n \Rightarrow \left|\sum_{j=n}^{m} a_j\right| < \varepsilon$
$\overset{2.24(3)}{\Rightarrow}$ $\sum_{j=1}^{\infty} a_j$ converges
↳ note that the opposite fails in general
(e.g. $\sum_{j=1}^{\infty} (-1)^j \frac{1}{j}$ converges, but not absolutely)

2.28 => Theorem: root test for absolutely convergence
Let $\sum_{j=1}^{\infty} a_j$ be a real or complex series
Let $L := \limsup_{j \to \infty} \sqrt[j]{|a_j|} = |a_j|^{\frac{1}{j}} \in \mathbb{R} \cup \{+\infty\}$
Then 1) if $L<1$, then $\sum_{j=1}^{\infty} a_j$ converges absolutely
2) if $L>2$, then $\sum_{j=1}^{\infty} a_j$ diverges
3) $L=1$ can have both cases

$q = L + \varepsilon$ for ε sufficient small

note that: $L<1 \overset{\text{def lim}}{\iff} q \in (0,1)$ and $N \in \mathbb{N}^*: (|a_j|)^{1/j} \leq q$ for all $j > N$
$L>1 \overset{2.22}{\iff} \exists$ subsequence $(a_{k_j})_{j \in \mathbb{N}^*}$ of (a_j) such that
$|a_{k_j}| \geq 1$ $\forall j \in \mathbb{N}^*$

↳ proof: by the above consideration, we have
$|a_j|^{1/j} = q \quad \forall j \geq N \Leftrightarrow |a_j| \leq q^j$
$\overset{(2)(3)}{\underset{2.25}{\Rightarrow}} \sum_{j=1}^{\infty} |a_j|$ converges

If $L > 0$, by above consideration:
$\overset{\text{div.}}{\underset{\text{test}}{\Rightarrow}} \sum_{j=1}^{\infty} a_j$ can not converge

2.29 ⇒ Theorem: Ratio test
Let a_j be a sequence of non-zero numbers
Let L be the $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$
1) If $L < 1$ then $\sum_{j=1}^{\infty} a_j$ converges absolutely (root test)
2) If $L > 2$, then $\sum_{j=1}^{\infty} a_j$ diverges

⇒ recall: $\sum_{n=1}^{\infty} a_n$ converges absolutely $\overset{\text{def.}}{\Leftrightarrow} \sum_{n=1}^{\infty} |a_n|$ converges
$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges
We shall see that $\exists (a_n)_{n\in\mathbb{N}^*}$ in $\mathbb{R}$: $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges
↳ Candidate example: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$, because $\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{n}\right|$ diverges

2.30 ⇒ Lemma: Let $a_1, \dots, a_m, b_1, \dots, b_m$ be real numbers, then for every $n \in \mathbb{N}^*$ $1 \leq n \leq m$:
$$\sum_{k=n}^{m} a_k b_k = a_m \sum_{k=n}^{m} b_k - \sum_{j=n}^{m-1} (a_{j+1} - a_j) \sum_{k=n}^{j} b_k \quad \text{(sum by parts)}$$
↳ proof: $n=1$; set $b_0 = 0$
$$\sum_{j=1}^{m-1} (a_{j+1} - a_j) \sum_{k=1}^{j} b_k = \underline{\sum_{j=1}^{m-1} a_{j+1} \sum_{k=1}^{j} b_k} - \sum_{j=1}^{m-1} a_j \sum_{k=1}^{j} b_k$$
$$= \sum_{j=2}^{m} a_j \sum_{k=1}^{j-1} b_k - \sum_{j=1}^{m-1} a_j \sum_{k=1}^{j} b_k$$
$$= \underline{\sum_{j=2}^{m} a_j \sum_{k=1}^{j} b_k} - \sum_{j=2}^{m} a_j b_j - \underline{\sum_{j=1}^{m-1} a_j \sum_{k=1}^{j} b_k}$$
$$= \underbrace{-a_1 \sum_{k=1}^{1} b_1}_{-a_1 b_1} + a_m \sum_{k=1}^{m} b_k - \sum_{j=2}^{m} a_j b_j$$
$$= a_m \cdot \sum_{k=1}^{m} b_k - \sum_{j=1}^{m} a_j b_j \Rightarrow \text{Statement for } n=1$$
for general $n \leq m$: set $a_1 = a_2 = \dots = a_{k-1} = b_1 = \dots = b_{k-1} = 0$

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2.31 => Theorem: Dirichlet

Let $(a_k)_{k\in\mathbb{N}^*}$ be a non-increasing sequence in $\mathbb{R}$, which converges to 0, and let $(b_k)_{k\in\mathbb{N}^*}$ be a sequence in $\mathbb{R}$ such that:
$\exists C>0 : \forall n\in\mathbb{N}^* \; |\sum_{k=1}^{n} b_k| < C$ (partial sums of $\sum_{k=1}^{\infty} b_k$ are bounded)
ex. $(-1)^k$ (not perse convergent)

Then $\sum_{k=1}^{\infty} a_k b_k$ converges (does not direct mean that abs. conv.)

↳ proof: let $\varepsilon>0$, since $a_k \xrightarrow{k\to\infty} 0$ and (a_k) non-decreasing
$\exists N\in\mathbb{N}^* : \forall k>N \; a_k < \varepsilon/C$, where C is the bound from the assumption on $\sum_{k=1}^{\infty} b_k$ *

2.32 => Example: The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges because setting $a_k = \frac{1}{k}$, $b_k = (-1)^k$, $k\in\mathbb{N}^*$ and apply theorem 2.31
Since a_k non-increasing and $|\sum_{k=1}^{n} b_k| < 2 \; \forall n$

* by triangle inq. $|\sum_{k=n}^{m} a_k b_k| \leq |a_m \sum_{k=n}^{m} b_k| + |\sum_{j=1}^{m-1} \overbrace{(a_{j+1} - a_j)}^{\leq 0} \sum_{k=n}^{j} b_k|$
for every $m \geq n \geq 1$
(Goal: $\ldots < \varepsilon$ for $m \geq n \geq \tilde{N}$)

Note that by assumption on $\sum_{k=1}^{\infty} b_k$, we have that

** $\sum_{k=n}^{m} b_k = \sum_{k=1}^{m} b_k - \sum_{k=1}^{n-1} b_k \qquad m \geq n > 1$

$\Rightarrow |\sum_{k=n}^{m} b_k| < C + C = 2C$

** $\leq a_m |\sum_{k=n}^{m} b_k| + \sum_{j=1}^{n-1} (a_j - a_{j+1}) |\sum_{k=n}^{j} b_k|$

$\leq a_m |\sum_{k=n}^{m} b_k| - a_m \sum_{k=n}^{m-1} b_k + C \cdot a_n$

$\sum_{k=n}^{m-1} (a_j - a_{j+1}) C = (a_n - a_m) C$
$= a_n C - a_m C$

$\sum_{j=n}^{m-1} (a_j - a_{j+1}) |\sum_{k=n}^{j} b_k|$

$= \sum_{j=n}^{m-2} (a_j - a_{j+1}) \underbrace{|\sum_{k=n}^{j} b_k|}_{<2C} + \overbrace{(a_{m-1} - a_m)}^{\geq 0} \overbrace{|\sum_{k=n}^{m-1} b_k|}^{<2C} < a_n 2C - \cancel{a_{m-1} 2C} + (\cancel{a_{m-1}} - a_m) 2C$

$\Rightarrow |\sum_{k=n}^{m} a_k b_k| < a_m \overbrace{|\sum_{k=n}^{m} b_k|}^{<2C} + 2a_n C - 2a_m C$

$< a_m 2C + 2a_n C - 2a_m C = 2a_n C$

$\Rightarrow$ for $m \geq n > N$: $2a_n C < {}^{2C}/_{C} \cdot \varepsilon \Rightarrow 2\varepsilon$

$\Rightarrow (\sum_{k=1}^{n} a_k b_k)_{n\in\mathbb{N}^*}$ is cauchy

Facts $\Rightarrow \sum_{k=1}^{\infty} a_k b_k$ converges

2.33 ⇒ Corrolary: Alternating series (Leibniz)
Let $(a_k)_{k\in\mathbb{N}^*}$ be a monotonic sequence converging to 0, then $\sum_{k=1}^{\infty} (-1)^k a_k$ converges

↳ proof: $b_k = (-1)^k$ satisfies the condition of bounded partial sums in 2.31 $\sum_{k=1}^{n} = \begin{cases} -1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

$\overset{2.31}{\Rightarrow}$ assertion, if (a_k) non-increasing
For $(a_k)_k$ non-decreasing: consider $\tilde{a}_k = -a_k, k\in\mathbb{N}^*$, $\tilde{b}_k = -b_k = (-1)^{k+1}, k\in\mathbb{N}^*$ and apply other case

2.35 ⇒ Theorem: Density test, cauchy
Let a_k be non-increasing and non-negatieve
Then the series $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{n=1}^{\infty} 2^n a_{2^n}$
$= (2\cdot a^2, 4\cdot a^4, 8\cdot a^8, \ldots)$
↳ proof: Zorich

2.34 ⇒ Example: $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p>1$ (we have already seen $p=1, p=2$)
Since $(\frac{1}{n^p})_{n\in\mathbb{N}^*}$ is non-increasing and positive. By 2.35
$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges (absolutely) $\Leftrightarrow \sum_{k=1}^{\infty} \frac{1}{(2^k)^p}\cdot 2^k$ converges
$= \sum_{k=1}^{\infty} 2^{-kp+k} = \sum_{k=1}^{\infty} 2^{k(1-p)}$ converges $\Leftrightarrow p>1$
(by ex. 2.20(1))

Chapter 3: Continuous functions

1. Limits of function " $\lim_{t \to x} f(t)$)

3.1 => **Definition:** (X,d) is a metric space, $D \subseteq X$
We call $x \in X$ a limit point of D (accumulation point) if:
$\exists (x_n)_{n \in \mathbb{N}^*}$ $D \setminus \{x\}$ such that $x_n \xrightarrow{n \to \infty} x$
$\{ \Leftrightarrow (x_n) \in (D\setminus\{x\}^{\mathbb{N}^*}) \}$ $\Leftrightarrow d(x_n, x) \xrightarrow{n\to\infty} 0$ (in $\mathbb{R}, d_2$)

3.2 => **Fact:** Note that x is a limit point of $D \Leftrightarrow \forall \varepsilon > 0$ the set
$\{y \in D\setminus\{x\}$ such that $d(x,y) < \varepsilon\}$ is non-empty
$\Leftrightarrow \{y \in D\setminus\{x\}$ such that $d(x,y) < \varepsilon\}$ is infinite

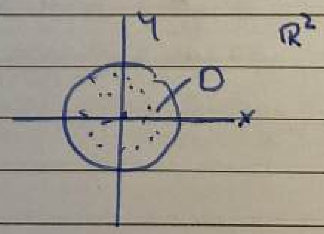
3.3 => **Examples:**

i) let $X = \mathbb{R}$, $d = d_2$, $D = (a,b)$, $a < b$
=> $[a,b] = \{x \in \mathbb{R} : x$ is limit point of $D\}$ 
↳ proof: given $x \in [a,b]$ define $x_n = x + \frac{1}{n}$ or
$x_n = x - \frac{1}{n}$ such that $x_n \in D$, for large enough n

ii) let (X,d) be any metric space, X non-empty.
Every finite set $D \subseteq X$ has no limit point.
↳ proof: Since x is limit point $\Leftrightarrow \exists (x_n)$ in $D\setminus\{x\}$:
$x_n \xrightarrow{n\to\infty} x$, by finiteness of D, the only convergent sequences
in $D\setminus\{x\}$ are eventually constant ($\exists N \in \mathbb{N}^*: x_n = c\ \forall n > N$ ($\in D\setminus\{x\}$))
=> D has no limit point
($X = \mathbb{R}$, $D = \{1,2\}$ has no limit point)

iii) X non-empty, $d = d_0$ discrete metric (see begin chapter 2):
no subset $D \subseteq X$ has a limit point
↳ proof: tutorial

iiii) $X = \mathbb{R}^p$, $p \in \mathbb{N}^*$, $d = d_2$ $(x,y) = \left(\sum_{i=1}^{d} |x_i - y_i|^2\right)^{1/2}$,
$D = \{x \in \mathbb{R}^p : d(x,0) < 1\}$
↳ for $p = 2$:
Limit point of D:
$D = \{x \in \mathbb{R}^2 : d(x,0) \le 1\}$



3.4 => **Definition**: Let $(X, d_X), (Y, d_Y)$ be metric spaces, $D \subseteq X$ and $f: D \to Y$

Then for a limit point $x \in X$ of D, an element $L \in Y$ is called "the limit of $f(t)$ as t goes to x" if:

$\forall \varepsilon > 0\ \exists \delta > 0 : \forall y \in D : (0 < d_X(x,y) < \delta \Rightarrow d_Y(L, f(y)) < \varepsilon)$

$(\Leftrightarrow x \neq y)$

Denoted by $L = \lim_{t \to x} f(t)$ $(\Leftrightarrow \forall y \in D \setminus \{x\} (d_X(x,y) < \delta \Rightarrow d_Y(L, f(y)) < \varepsilon))$

↳ If $X = \mathbb{R}$, $d_X = d_2$ we define the following variants of the above:

- for a limit point $x \in \mathbb{R}$ of $D \cap (x, \infty)$, the limit $\lim_{t \to x} f|_{D \cap (x,\infty)}(t)$, if exists, is called the right-sided limit of $f(t)$ as t goes to x, with symbol $\lim_{\substack{t \to x^+ \\ (t \in D)}} f(t)$ (or $\lim_{t \downarrow x} f(t)$)

  $f: D \to Y$
  $f|_{D \cap (x,\infty)} \to Y, z \to f(z)$

- for a limit point $x \in \mathbb{R}$ of $D \cap (-\infty, x)$ the limit $\lim_{t \to x} f|_{D \cap (-\infty, x)}(t)$, if exist, is called the left-sided limit of $f(t)$ as t goes to x, with symbol $\lim_{t \to x^-}$

- If D is not bounded above/below. We call $L \in Y$ the limit of $f(t)$ as t goes to $+\infty / -\infty$, if

  $\exists h \in \mathbb{R} : (h, \infty) \subseteq D\ \forall \varepsilon > 0\ \exists c \in \mathbb{R}\ \forall y > c\ d_Y(L, f(y)) < \varepsilon$

  $\exists h \in \mathbb{R} : (-\infty, h) \subseteq D\ \forall \varepsilon > 0\ \exists c \in \mathbb{R}\ \forall y < c\ d_Y(L, f(y)) < \varepsilon$

- If $Y = \mathbb{R}$, $d_Y = d_2$ we can define $\lim_{t \to x} f(t) = \pm \infty$

  ↳ proof: tutorial (ex. $\lim_{x \to 0} \frac{1}{x} = \infty$)

3.5 => Facts: let $(X, d_X), (Y, d_Y)$ be metric spaces, $D \subseteq X$, $f: D \to X$

1) If x limit point of D: $\lim_{t \to x} f(t) = L \in Y$

   $\Leftrightarrow$ $\forall$ sequences $(x_n)_{n \in \mathbb{N}^*}$ in $D \setminus \{x\}$ converging to x holds that $(f(x_n))_{n \in \mathbb{N}^*}$ converges to L (in Y).

   This statement remains true for right-/left-sided limits, limits to $+/-\infty$ with the corresponding definitions for sequences

   => we can use a lot of the properties of limits of the sequences from chapter 2

2) if $Y = \mathbb{R}$, $d_Y = d_2$ and $g: D \to \mathbb{R}$ then

$0 \leq f(t) \leq g(t) \ \forall t \in D$

$\overset{(1)}{\Rightarrow} 0 \leq \lim_{t \to x} f(t) \leq \lim_{t \to x} g(t)$, if limits exists

if $\lim_{t \to x} g(t) = 0$, then $\lim_{t \to x} f(t)$ exists and equals 0 ((1), sqeeze theorem)

3) If $Y = \mathbb{R}$, $d_Y = d_2$, if $g: D \to Y$ and $\lim_{t \to x} f(t)$, $\lim_{t \to x} g(t)$ exists, then also

$\lim_{t \to x} (f+g)(t)$ exists and equals $\lim_{t \to x} f(t) + \lim_{t \to x} g(t)$

$\lim_{t \to x} (f \cdot g)(t)$ exists and equals $\lim_{t \to x} f(t) \cdot \lim_{t \to x} g(t)$

$\lim_{t \to x} \frac{f(t)}{g(t)}$ exists and equals $\frac{\lim f(t)}{\lim g(t)}$ (if $\lim_{t \to x} g(t) \neq 0$)

(in the case $Y = \mathbb{R}^p$, at least the "+" case can be recovered)

4) The above statements have analagous meaning for one-sided limits and limits involving $\pm\infty$

3.6 ⇒ **Definition:** (X,d) metric space, $D \subseteq X$. We call $x \in D$ an isolated point of D if x is not a limit point

3.7 ⇒ Examples:

i) $X = Y = \mathbb{R}$, $d_X = d_Y = d_2$, $f: X \xrightarrow{\mathbb{R} \to \mathbb{R}} Y$, $f(x) = xk + m$, $x \in \mathbb{R}$ $k, m \in \mathbb{R}$

$\forall x \in \mathbb{R}$: $\lim_{t \to x} f(t) = xk + m \ (= f(x))$ because by definition of limit:

$\forall \varepsilon > 0 \ \exists \delta = \varepsilon / k > 0 : \forall y \in \mathbb{R} \ 0 < |x - y| < \delta \Rightarrow |\underbrace{kx}_{L} - \underbrace{ky}_{f(y)}| < \varepsilon$

ii) $X = Y$ is non-empty, $d_X = d_Y = d_0$, $f: X \to X$

$\lim_{t \to x} f(t) = ?$ tutorial

2. Continuous functions

3.8 ⇒ **Definition:** let (X, d_X), (Y, d_Y) be metric spaces. Then f is called continuous at $x \in X$, if

$\forall y \in X : (d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon)$ $(\delta_{\varepsilon, x})$

↳ We call f continuous if f is continuous at x $\forall x \in X$

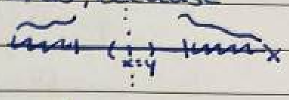
$\forall x \in X \ \forall \varepsilon > 0 \ \exists \delta \ \forall y \in X \ (d_X(x,y)) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon)$

↳ If $\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in X \ \forall y \in X (d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon)$ then f is called uniformily continuous (on X) (δ does not depend on X)

3.9 $\Rightarrow$ Facts:

1) How does continuity relate to limits?
If $D \subseteq X$, $x \in X$ limit point of D, we define $\lim_{t \to x} f(t)$
in the case that f is continuous at $x \overset{\text{def lim}}{\underset{\text{con.}}{\Rightarrow}} \lim_{t \to x} f(t) = f(x)$
Conversely if x limit point of X and $\lim_{x \to t} f(t) = f(x)$, then also f is continuous at x

2) For x not limit point of X: f is always continuous, because
$\nexists$ sequence in $X \setminus \{x\}$ converging to x



$\Rightarrow \exists \delta > 0: \{y \in X: 0 < d(x,y) < \delta\}$ is empty-set
$\Rightarrow \forall \varepsilon > 0 \; \exists \delta \; d(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$ is always true

3) f is continuous at $x \Leftrightarrow \forall (x_n)_{n \in \mathbb{N}}$ in X converging to x
it follows that $(f(x_n))_{n \in \mathbb{N}}$ converges to $f(x)$

4) "D vs X" we can also define continuity for function $f: D \to Y$
where $D \subseteq X$. This can be done by recalling that $(D, d|_{D \times D})$
metric space (and replace (X, d_X) by $(D, d|_{D \times D})$ in def 3.8)
$\Rightarrow$ Continuity is a local property
$\Leftrightarrow$ if $\exists r > 0$, $g: \{y \in X: d(x,y) < r\} \to Y$: f continuous at x
$\Leftrightarrow$ g continuous at x, $f(y) = g(y) \; \forall y \in M$

5) if $(Y, d_Y) = (\mathbb{R}, d_2)$ $f: X \to \mathbb{R}$ are continuous at $x \in X$
$\Rightarrow f+g, f \cdot g$ are continuous at x

6) if $g(y) \neq 0 \; \forall y \in X \Rightarrow \frac{f}{g}$ continuous at x

7) $f: D \subseteq \mathbb{R} \to \mathbb{R}$, $x \mapsto x^m$, $m \in \mathbb{N}$ are continuous
Every polynominal (with coëfficients in $\mathbb{R}$) continuous
For polynominals p, q the function $f = \frac{p}{q}$ is continuous on
$D \subseteq \mathbb{R}$ for function q having no zero in D

3.10 => proposition: Composition of continuous functions

Let (X,d_X), (Y,d_Y), (Z,d_Z) be metric spaces and $f: X \to Y$, $g: Y \to Z$ such that f is continuous at $x \in X$, and g is continuous at $f(x)$

Then $g \circ f : X \to Z$ is continuous at x

↳ proof:

Option 1: use sequences

To show $\forall (x_n)$ in X converges to $x \overset{!}{\Rightarrow} (g \circ f)(x_n)$ converges to $(g \circ f)(x)$

Since f is continuous at x, by facts 3.9(3), $f(x_n) \xrightarrow{n \to \infty} f(x)$

and again using this fact and g is continuous at $f(x)$

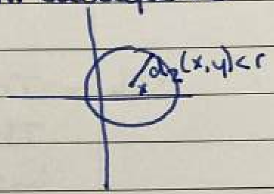
$\Rightarrow g(f(x_n)) \xrightarrow{n \to \infty} g(f(x))$

Option 2: By definition of continuity (Tutorial)

3.11 => Definitions: Let (X,d) be metric space, $M \subseteq X$. We call

- M closed if M contains all it's limit point
- M compact if every sequence in M has a convergent subsequence with limits in M
- M is bounded if $\exists x \in X, \exists r > 0 : M \subseteq \{y \in X : d(x,y) < r\}$

(recall how we defined bounded sequences in metric space def 2.5)



3.12 => Examples:

1) $X = \mathbb{R}$, $d_X = d_2$, $M = [a,b]$ for $a \leq b \in \mathbb{R}$ is closed

Let (x_n) in $[a,b] \Leftrightarrow \forall n \in \mathbb{N}^* : a \leq x_n \leq b$ if $x_n \xrightarrow{n \to \infty} x$ then $a \leq x \leq b$ (by facts limits on sequences)

2) Same as (1), $M = (a,b)$ is not closed

Take $x_n = \begin{cases} a + \frac{1}{n} & n > N \\ \frac{a+b}{2} & n \leq N \end{cases}$ (since $\frac{1}{n} \xrightarrow{n \to \infty} 0$ $\exists N \in \mathbb{N}^* : a + \frac{1}{n} < b$ $\forall n > N$)

$\Rightarrow x_n \in (a,b)$ $\forall n \in \mathbb{N}^*$ but $\lim_{n \to \infty} x_n = a \notin (a,b) \Rightarrow (a,b)$ not closed

3) Same as (1), $M = [a, \infty)$ is closed (as in (1) but M is not bounded)

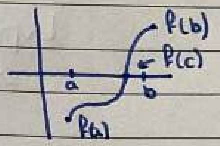
$\Rightarrow M$ is not compact, because $x_n = a + n$, $n \in \mathbb{N}^*$ has no convergent subsequence ($|x_n - x_m| \geq 1$ $\forall n \neq m \Rightarrow$ no subsequence is cauchy)

3.13 $\Rightarrow$ Theorem: let (X,d) be a metric space and $M \subseteq X$. Then
1) M compact $\Rightarrow$ M is closed and bounded
↳ proof: If M is compact every (convergent) sequence in M has a convergent subsequence with limits in M.
But since the limit of a subsequence equals the limit of the sequence (facts on subsequences)
$\Rightarrow$ Every limit point lies in M
Bounded: Hint: suppose M is not bounded and conclude M is not compact
2) If $X = \mathbb{R}^p$, $d = d_2$, the converse also holds ($p \in \mathbb{N}^*$) (Bolzano Weierstrass)
↳ Given $(x_n)_{n\in\mathbb{N}^*}$ in M, need to find convergent subsequence (with limit in M)
Since M is bounded, the sequence is bounded
$\overset{\text{Bolz. Wei.}}{\Rightarrow}$ $\exists$ subsequence that converges to limit $x \in X$
$\overset{\text{m closed}}{\Rightarrow}$ $x \in M$

3.14 $\Rightarrow$ Theorem: Let (X,d_x) (Y,d_y) be metric spaces $f: X \to Y$ continuous.
Then $M \subseteq X$ compact $\Rightarrow$ $f(M)$ is compact too (in Y)
↳ proof: tutorial

3.15 $\Rightarrow$ Corrolary: Let (X,d) be a metric space, $f: X \to \mathbb{R}$ continuous,
where $\mathbb{R} = (\mathbb{R}, d_2)$, $K \subseteq X$ compact
$\Rightarrow f(K) = \{z \in \mathbb{R} : \exists x \in K \; f(x) = z\}$ has a maximum and minimum
In particular, $\exists x_{min} \in K, x_{max} \in K$ $f(x_{min}) = \min f(K)$, $f(x_{max}) = \max f(x)$
↳ proof: By previous theorem, $f(K)$ is compact. Since $f(K) \subseteq \mathbb{R}$, $f(K)$ has a sup and inf which are finite, by boundedness of $f(x)$
For S being the sup $f(K)$, $\exists$ seq (y_n) in $f(K)$ such that $y_n \xrightarrow{n\to\infty} S$
$\Rightarrow \exists (x_n)$ in K $\underbrace{f(x_n)}_{=y_n} \xrightarrow{n\to\infty} S$
$\overset{K\text{ compact}}{\Rightarrow}$ (x_n) has a convergent subsequence with limit in K, call it X
$\overset{f\text{ cont.}}{\Rightarrow}$ $f(x_n) \xrightarrow{n\to\infty} f(x) = S$ Let $x_{max} = X$
Similarly for infimum

3.16 $\Rightarrow$ Theorem: Intermediate value theorem I



Let $I[a,b]$, with $a<b$ real numbers
and $f: [a,b] \to \mathbb{R}$ continuous
If $f(a)\cdot f(b)<0$, then $\exists c \in [a,b] : f(c)=0$
↳ proof: Consider $\frac{a+b}{2} \in (a,b)$ (by $a<b$)
if $f(\frac{a+b}{2})=0$, done let $c=\frac{a+b}{2}$
Otherwise define $a' = \frac{a+b}{2}$ or $b' = \frac{a+b}{2}$ depending on whether
(2) $f(a)f(\frac{a+b}{2})<0 \vee$ (1) $f(b)f(\frac{a+b}{2})<0$, because $f(a)f(b)<0$
and $\begin{cases} b'=b, 1 \\ a'=a, 2 \end{cases}$ now repeat
↳ There are 2 cases:
1) Computer scientist case:
$\exists c$ being the midpoint of the previous interval such that $f(c)=0$
2) mathematicians case:
We don't have the above case. Then we constructed a sequence of nested intervals $[a_n, b_n]_{n\in\mathbb{N}^*}$ with $b_n - a_n \to 0$ $(n\to\infty)$
(because $b_{n+1}-a_{n+1} = \frac{b_n - a_n}{2}$ $\forall n \in \mathbb{N}^*$)
$\overset{\text{nested interval}}{\Rightarrow}$ $\exists! c \in [a,b] : c \in \bigcap_{n\in\mathbb{N}^*} [a_n, b_n]$
$\Rightarrow$ $f(c)=0$ because $f(a_n)\cdot f(b_n)<0$ $\forall n \in \mathbb{N}^*$ and $\begin{cases} a_n \to c \\ b_n \to c \end{cases}$ $(n\to\infty)$
$\overset{\text{cont.}}{\Rightarrow}$ $\begin{matrix} f(a_n)\to f(c) \\ f(b_n)\to f(c) \end{matrix}$ $\Rightarrow \underbrace{f(a_n)\cdot f(b_n)}_{<0} \to f(c)^2 \geq 0$
$\Rightarrow f(c)^2 \geq 0 \wedge f(c)^2 \leq 0$
$\Rightarrow f(c)=0$

3.17 $\Rightarrow$ Corollary: Intermediate value theorem II
Let $I[a,b]$, with $a<b$ real numbers and $f: [a,b] \to \mathbb{R}$ continuous
$\forall d \in [f(a), f(b)] : \exists c \in [a,b] : f(c)=d$
(where $[f(a), f(b)]$ refers to $[f(b), f(a)]$ if $f(b)<f(a)$)

3.18 $\Rightarrow$ Example: let $f: \mathbb{R}\to\mathbb{R} : t \mapsto t^n$, $n\in\mathbb{N}^*$ then $f([0,\infty)) = [0,\infty)$
by corollary 3.17 (applied on $[m-1, m]$ $m\in\mathbb{N}^*$

3.19 $\Rightarrow$ Theorem: Uniform continuity for functions on compact sets
Let X, Y be metric spaces $f: X\to Y$ continuous. If X compact, then f is uniformly continuous

↳ proof: Suppose f is not uniformly continuous

$\Rightarrow \exists \varepsilon>0\ \forall \delta>0\ \exists x,y\in X: d_X(x,y)<\delta \not\Rightarrow d_Y(f(x),f(y))<\varepsilon$

$\Leftrightarrow d_X(x,y)<\delta \Rightarrow d_Y f(x),f(y)) \geq \varepsilon$

$\Rightarrow \exists \varepsilon>0 : \exists$ seq $(x_n),(y_n)$ in $X : d(x_n,y_n)<\frac{1}{n}\ \forall n\in\mathbb{N}^* \wedge d(f(x_n),f(y_n))\geq\varepsilon$

Since X is compact $\exists (x_{n_k})_{k\in\mathbb{N}^*}$ Subsequence of $(x_n)_{n\in\mathbb{N}^*}$ such that $x_{n_k} \xrightarrow{n\to\infty} x\ (k\to\infty)$. Again since X is compact the sequence $(y_{n_k})_{k\in\mathbb{N}^*}$ has a convergent subsequence $(y_{n_{k(l)}})_{l\in\mathbb{N}^*}$

$\Rightarrow (x_{n_{k(l)}})_{l\in\mathbb{N}^*}$ and $(y_{n_{k(l)}})_{l\in\mathbb{N}^*}$ both converge to $\overset{\in}{x}$ and $\overset{y}{y}$ respec.

$\Rightarrow d_X(x,y)=0$ (Δ-inequality) $\Leftrightarrow x=y \Rightarrow d(f(x),f(y))=0$ ↯

$=f(x),f(x))$

3. Sequence of functions

$\Rightarrow$ Recall: Definition of sequences

function mapping $\mathbb{N}^*$ to X, where (X,d) is a metric space

↳ So far mostly $X=\mathbb{R}$, $d=d_2$

↳ Now choose $X=\{f: I\to \underset{\mathbb{C}}{\mathbb{R}} \mid f$ is continuous / bounded$\}$ $I\subseteq\mathbb{R}$ intervals

$\overset{def}{=} C(I)$

3.20 $\Rightarrow$ proposistion: The function $d_\infty : C(I)\times C(I)\to[0,\infty)$,

$d_\infty(f,g)=\sup_{t\in I}|f(t)-g(t)|$ defines a metric an $C(I)$

$(C(I),d_\infty)$ is a metric space

↳ proof: chech conditions of a metric

1) $d_\infty(f,f)\overset{def}{=}\sup_{t\in I}|f(t)-f(t)|=0$

and $d_\infty(f,g)=0$ for $f,g\in C(I) \Rightarrow \sup_{t\in I}|f(t)-g(t)|=0$

$\Rightarrow |f(t)-g(t)|\leq 0\ \forall t\in I \Rightarrow f(t)=g(t)\ \forall t\in I \Leftrightarrow f=g$

2) $d_\infty(f,g)=d_\infty(g,f)\ \forall f,g\in C(I)$

3) $d_\infty(f,g)+d_\infty(g,h)=\sup_{t\in I}|f(t)-g(t)|+\sup_{t\in I}|g(t)-h(t)|\geq$

$\sup_{t\in I}|f(t)-h(t)|=d_\infty(f,h)$

$\Rightarrow$ let $h: X\to Y$ be continuous and $K\subseteq X$ compact

$\Rightarrow h(K)$ is compact

If $K\subseteq X$ is closed $\not\Rightarrow h(K)$ is closed

3.21 => Facts

- If I compact then $d_\infty(f,g) = \max_{t\in I} \underbrace{|f(t)-g(t)|}_{=h(t)}$
  $h: I \to \mathbb{R}$ continuous (as composition of continuous functions)
  As consequence: $h(t)$ compact
  $\Rightarrow \sup_{t\in I} = \max_{t\in I} h(t)$

- $\|.\|: X \to \mathbb{R}$, X-vector space such that $\forall x,y \in X, c\in\mathbb{R}$

$$\left\{\begin{array}{ll} 1 & \|x\| = 0 \Leftrightarrow x = 0 \in X \\ 2 & \|cx\| = \| = |c|\|x\| \\ 3 & \|x+y\| \leq \|x\| + \|y\| \end{array}\right\} \|.\| \text{ is called a norm on } X$$

The pair $(X, \|.\|)$ defines a metric space $(X, d_{\|.\|})$ where
$d_{\|.\|}(x,y) = \|x-y\| \quad \forall x,y \in X$
($X = \mathbb{R}, \|.\| = |.| \Rightarrow d_{\|.\|} = d_2$) normed spaces are special metric spaces
By facts on continuous functions, $C(I)$ is a vector space
For $\|f\|_\infty = \sup_{t\in I} |f(t)|$ we have a norm such that
$d_\infty(f,g) = \|f-g\|_\infty \Rightarrow (C(I), \|.\|_\infty)$ is a normed space

- We could more generally define $C_b(X)$ where (X,d) is a metric space, $C_b(X) = \{ f: X \to \mathbb{R} \mid f \text{ continuous and bounded} \}$
Then $(C_b(X), d_\infty)$, with $d_\infty(f,g) = \sup_{t\in X} |f(t)-g(t)|$ is a metric space (proposition 3.20 generalized)

3.22 => **Definition**: Uniform convergence & pointwise convergence
Let E be a set, $f_n: E \to \mathbb{R}, n\in\mathbb{N}^*$, $f: E \to \mathbb{R}$, then we say that $(f_n)_{n\in\mathbb{N}^*}$

i) Converges uniformly to f if: (n does not depend on x)
$\forall \varepsilon > 0\ \exists N \in \mathbb{N}^*\ \forall x \in E\ \forall n \geq N: |f_n(x) - f(x)| < \varepsilon$

ii) Converges pointwise to f if: (n does depend on x)
$\forall x \in E\ \forall \varepsilon > 0\ \exists N \in \mathbb{N}^*: \forall n \geq N\ |f_n(x) - f(x)| < \varepsilon$

3.23 => Facts:

- (f_n) converges uniformly to $f \Leftrightarrow \sup_{x\in E} |f_n(x) - f(x)| \xrightarrow{\text{in } (\mathbb{R}, d_2)} 0 \ (n\to\infty)$
  if $f_n, f \in E$ $E \subseteq \mathbb{R}$ $\Leftrightarrow d_\infty(f_n, f) \to 0 \ (n\to\infty)$
  $\Leftrightarrow f_n \to f$ in $(C(E), d_\infty)$

=> For continuous and bounded functions on $E \to \mathbb{R}$, uniform convergence is the same to say that the sequence converges in $(C(E), d_\infty)$

- Uniform convergence $\Rightarrow$ pointwise convergence
  The converse is not true in general

- In examples/applications/exam questions we typically know more about the f_n's and would like to conclude something on the limit f (uniform convergence)
  For example: Suppose f_n continuous for all n, and $f_n \to f$ uniformly. Is f continuous?

3.24 $\Rightarrow$ Theorem: Let $I \subseteq \mathbb{R}$, $(f_n)_{n\in\mathbb{N}^*}$ be a sequence of continuous functions from I to $\mathbb{R}$.
If (f_n) converges uniformly to $f: I \to \mathbb{R}$, then f is continuous
↳ proof: Let $x \in I$, $y \in I$

$$|f(x) - f(y)| = |\lim_{n\to\infty} f_n(x) - \lim_{n\to\infty} f_n(y)|$$
$$= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)|$$
$$\text{Δ-inequal} \leq \underbrace{|f(x) - f_n(x)|}_{<\varepsilon/3} + \underbrace{|f_n(x) - f_n(y)|}_{<\varepsilon/3} + \underbrace{|f_n(y) - f(y)|}_{<\varepsilon/3}$$

by uniform convergence: $< \varepsilon/3$ $\quad$ $< \varepsilon/3$ $\quad$ $< \varepsilon/3$
Let $\varepsilon > 0$ $\exists N \in \mathbb{N}^*$ $\forall n \geq N$ $\quad$ ⇑
(f_n) continuous at $x \in I \Rightarrow \exists \delta > 0: \forall y \in I: d_2(x,y) < \delta$
$\Rightarrow$ let $x \in I$ and $\varepsilon > 0$: $(\exists N \in \mathbb{N}^*\ \forall y \in I\ |f(y) - f_n(y)| < \varepsilon/3)$
$\exists \delta > 0$ (from f_n): $(\forall y \in I: d_2(x,y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon/3)$
$\forall y \in I\ d_2(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon\ (= \varepsilon/3 + \varepsilon/3 + \varepsilon/3)$

3.25 $\Rightarrow$ Example: Let $f_n: [0,q] \to$ $q > 0$, $f_n(x) = x^n$, $x \in [0,q]$

for $x \in [0,q]$ $f_n(x) \to \begin{cases} 0, & q < 1 \\ 1, & q = 1 \\ +\infty, & q > 1 \end{cases}$ (f_n) converges pointwise on $[0,1]$

Uniform convergence by fact 3.23 (f_n) uniformly converges to
$f(x) \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases} \Leftrightarrow \sup_{x\in[0,1]} |f_n(x) - f(x)| \to 0\ (n\to\infty)$
$= \sup_{x\in[0,1)} |f_n(x) = \sup_{x\in[0,1)} |x^n| = 1\ \forall n$

$\Rightarrow (f_n)$ does not converge uniformly on $[0,1]$
Let's now consider $(f_n)_n$ on $[0,q]$ for $q < 1$
Claim: (f_n) converges uniformly to 0 on $[0,q]$
↳ proof: $\sup_{x\in[0,q]} |f(x) - 0| = \sup_{x\in[0,q]} x^n = q^n \to 0\ (n\to\infty$

3.26 $\Rightarrow$ proposistion: Let $I \subseteq \mathbb{R}$ interval, $(f_n)_{n\in\mathbb{N}^*}$

Then $(f_n)_{n\in\mathbb{N}^*}$ converges in $(C(I), d_\infty)$ (uniformly)

$\Leftrightarrow$ $(f_n)_{n\in\mathbb{N}^*}$ is a cauchy sequence in $(C(I), d_\infty)$

↳ proof:

(f_n) cauchy in $(C(I), d_\infty) \Leftrightarrow \forall \varepsilon>0\ \exists N\in\mathbb{N}^*\ \forall n,m\geq N\ \underline{d_\infty(f_n, f_m) < \varepsilon}$

$= \sup_{t\in I} |f_n(t) - f_m(t)| < \varepsilon$

$\Leftrightarrow |f_n(t) - f_m(t)| < \varepsilon$

$\Rightarrow \forall t\in I\ (f_n(t))_{n\in\mathbb{N}^*}$ is cauchy in $(\mathbb{R}, d_2)$ Call the Limit $f(t)$.

Rest of the proof in the lecture notes

3.27 $\Rightarrow$ proposistion: Let (f_n) be a sequence of functions from $E \to \mathbb{R}$ (E set)

then $\sum_{n=1}^{\infty} f_n$ converges to $f: E \to \mathbb{R}$ uniformly

$\Leftrightarrow$ $(\sum_{n=1}^{\infty} f_n)$ converges to f uniformly ?

If $(f_n) \in C(E)$, $E \subseteq \mathbb{R}$ $\forall n\in\mathbb{N}^*$ then

$\sum_{n=1}^{\infty} \|f_n\|_\infty < \infty$ (converges) $\Rightarrow \sum_{n=1}^{\infty} f_n$ converges uniformly on E

(even $\sum_{n=1}^{\infty} |f_n(.)|$ converges uniformly)

↳ proof: See ps 5 (Weierstraß test)