Final Exam: Analysis-1 (202200143), MOD-01-AM: Structures and Models

Instructors: Pranab Mandal and Carlos Perez Arancibia

Date/Time: 08-November-2023. 13:45 – 16:45

- Closed book exam! May use one single-sided handwritten A4-paper.
- All answers must be motivated. including the answers of Section C.
- Answers for Section A *must* use the four steps (practised during Tutor Sessions).
  - (i.) Get Started: describe what the problem is about and your initial thoughts
  - (ii.) Devise Plan: provide an outline how you plan to solve (or have solved) the problem
  - (iii.) Execute: execute your plan (and try) to reach your solution
  - (iv.) Evaluate: reflect on your solution and/or approach

  Points are distributed (roughly) as: steps (i.)+(ii.) 35%. step (iii.) 50% and step (iv.) 15%.
- Section Grade: $\dfrac{\text{Obtained score}}{\text{Total points}} \times 9 + 1$ (rounded off to one decimal place)
- Course Grade: According to the assessment scheme (published elsewhere)
- Good Luck!

Section C:

Total Points : 15

1. Answer the following questions for the polynomial $p(\cdot)$ given by [3+3]

$$p(x) = x^6 - 3x^5 + 2x^4 + x^3 - 3x^2 + 2x.$$

   (a) Find all the real and complex roots of p.
   [Hint: Note that $p(2) = 0$.]

   (b) Show that there exist at least two distinct values c_1 and c_2 on the interval $(0, 2)$ such that $p'(c_1) = p'(c_2) = 0$.

2. Answer the following questions for the function $f : \mathbb{R} \to \mathbb{R}$ defined as [1+2+3+3]

$$f(x) = \begin{cases} \dfrac{\sin(x \ln x)}{\ln x} & \text{if } x \in (0, 1) \cup (1, \infty), \\ a & \text{if } x \in \{1\}, \\ b + cx + dx^2 & \text{if } x \in (-\infty, 0], \end{cases}$$

   where $a, b, c, d \in \mathbb{R}$.

   (a) For which values of a is $f(x)$ continuous at $x = 1$?

   (b) For which values of b, c, and d is $f(x)$ continuous at $x = 0$?

   (c) For which values of b, c, and d is $f(x)$ differentiable at $x = 0$?

   (d) Are there values of b, c, and d for which $f''(0)$ exists?

Section A: [Follow the four-step proecdure] Total Points : 35

3. Let E be a bounded non-empty set of real numbers and $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$. Define

$$A := \{\alpha\, x + \beta : x \in E\}.$$

Prove that $\sup(A) = \alpha \sup(E) + \beta$. [2+3+1]

4. Prove using the definition (of the limit of a sequence) that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$. [4+5+1]

[Hint: The following inequality may be helpful. However, prove it if you use it.

$$n \equiv (1 + a_n)^n \geq \frac{n(n-1)}{2}\, a_n^2, \quad \text{where } a_n := n^{\frac{1}{n}} - 1.$$
]

5. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $a \in \mathbb{R}$ be such that $f(a) > 0$. Prove that there exists an open interval. I, containing a such that $f(x) > 0$ for all $x \in I$. [3+3+1]

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be 4-times differentiable with $f(0) = f'(0) = f''(0) = 0$ and $f'''(0) = 2$. Show that there exists an open interval around 0 such that f is strictly increasing on that interval. [5+6+1]

[Hint: The following intermediate steps can be useful:

- Show using, among others, the mean value theorem that there is an open interval I containing 0 such that for $x \in I$ it holds that

$$f''(x) > 0 \quad \text{if } x > 0 \qquad \text{and} \qquad f''(x) < 0 \quad \text{if } x < 0.$$

- Show that $f'(x) > 0$ for all $x \in I$ $(x \neq 0)$.
]