Exam Limits to Computing (201300042)

Thursday, November 4, 2021, 8:45 – 11:45

- You can bring printouts of the sheets, lecture notes, exercises, solutions (mine and yours) to the exam or anything else printed or written on paper.
- Electronic devices of any kind are not allowed.
- This exam consists of four problems.
- Please start a new page for each problem.
- The total number of points is 50. In addition, you can get 5 bonus points from Exercise 1(d). Sufficient for passing are 25 points.

1. Computability

Let

$$\textsc{ImageH} = \{g \in \mathcal{G} \mid g \in \mathrm{im}(\varphi_g)\}.$$

(a) *(8 points)* Is $\textsc{ImageH} \in \mathsf{RE}$?

(b) *(8 points)* Is $\textsc{ImageH} \in \mathsf{co\text{-}RE}$?

(c) *(2 points)* Is $\textsc{ImageH} \in \mathsf{REC}$?

(d) *(5 bonus points)* Prove that $\textsc{ImageH}$ is not an index set.

2. NP-Completeness

A Hamiltonian cycle of an undirected graph $G = (V, E)$ is a simple cycle of G that contains every vertex of G exactly once. Let

$$\text{HAMCYCLE} = \{G \mid G \text{ contains a Hamiltonian cycle}\}.$$

A simple path in an undirected graph $G = (V, E)$ is a sequence $(v_0, v_1, \dots, v_k)$ of *distinct* vertices with $\{v_{i-1}, v_i\} \in E$ for all $i \in \{1, \dots, k\}$. The length of such a path is k, i.e., the number of edges that it contains. Let

$$\text{LONGPATH} = \{(G, k) \mid G \text{ contains a simple path of length } k\}.$$

(8 points) Prove that LONGPATH is NP-complete.

Hint: You can use the fact that HAMCYCLE is NP-complete.

3. Complexity Classes

Let $\mathsf{E} = \mathsf{DTime}(2^{O(n)})$. Recall that $\mathsf{EXP} = \bigcup_{c>0} \mathsf{DTime}(2^{O(n^c)})$.

(a) *(2 points)* Prove that $\mathsf{E} \subsetneq \mathsf{EXP}$.

(b) *(7 points)* Prove that E is not closed under polynomial-time many-one reductions. This means that there are problems $A \notin \mathsf{E}$ and $B \in \mathsf{E}$ with $A \leq_{\mathrm{P}} B$.

(c) *(3 points)* Prove that $\mathsf{E} \neq \mathsf{PSPACE}$.

4. Questions

Are the following statements true or false? Justify your answers briefly.

(a) *(2 points)* We have $\mathsf{L} \subseteq \mathsf{P} \subseteq \mathsf{PSPACE}$, and at least one of the inclusions is strict.

(b) *(2 points)* For all $L \subseteq \mathbb{N}$, the following holds: If χ_L is total, then $L \in \mathsf{REC}$.

(c) *(2 points)* For all $L \subseteq \mathbb{N}$, we have $L \in \mathsf{co\text{-}RE}$ if and only if $\overline{L} \in \mathsf{RE}$.

(d) *(2 points)* If $L \leq H_0$, then L is recursively enumerable.

(e) *(2 points)* If $\mathsf{P} = \mathsf{NP}$, then 2SAT is NP-complete.

(f) *(2 points)* If 2SAT is NP-complete, then $\mathsf{P} = \mathsf{NP}$.