Exam Limits to Computing (201300042)

Thursday, November 2, 2017, 8:45 – 11:45

- You can bring printouts of the sheets, lecture notes, exercises, solutions (mine and yours) to the exam or anything else printed or written on paper.
- Books and electronic devices of any kind are not allowed.
- This exam consists of five problems.
- Please start a new page for each problem.
- The total number of points is 63+7 = 70. The distribution of points is according to the following table.

1: 12	2a: 6	3: 10	4a: 4	5a: 3
	2b: 7		4b: 6	5b: 3
	2c: 3		4c: 4	5c: 5

1. NP-Completeness

An instance of SETCOVER is a finite set U together with subsets $S_1, \dots, S_m \subseteq U$ and a $k \in \mathbb{N}$. The question is if there exists k of these subsets that cover all elements in U.

More formally, an instance as described above is a "yes" instance if there exists a set $I \subseteq \{1, \dots, m\}$ with $|I| \leq k$ such that

$$\bigcup_{i \in I} S_i = U.$$

(12 points) Prove that SETCOVER is NP-complete.

Hint: VERTEXCOVER $= \{(G, k) \mid$ undirected graph G has a vertex cover of size $k\}$ is NP-complete.

2. Decidability and Recursive Enumerability

We call a (partial) function $f : \mathbb{N} \to \mathbb{N}$ *weakly increasing* if it satisfies the following property: For all $n \in \mathbb{N}$ with $n, n+1 \in \text{dom}(f)$, we have $f(n) \leq f(n+1)$.

Example: The following function $f : \mathbb{N} \to \mathbb{N}$ is weakly increasing:

$$n \mapsto f(n) \begin{cases} = 100 & \text{if } n < 10, \\ = n & \text{if } 10 < n < 20, \\ = 0 & \text{if } 20 < n, \text{ and} \\ \text{undefined} & \text{if } n = 10 \text{ or } n = 20. \end{cases}$$

Consider the following decision problem:

$$\text{WEAKINC} = \{g \in \mathcal{G} \mid \varphi_g \text{ is weakly increasing}\}.$$

(a) *(6 points)* Is WEAKINC $\in$ REC? Prove your answer.

(b) *(7 points)* Prove that WEAKINC $\in$ co-RE.

(c) *(3 points)* Is WEAKINC $\in$ RE? Prove your answer.

3. Logarithmic Space

For a decision problem $A \subseteq \{0,1\}^\star$, let

$$A^\star = \{x \in \{0,1\}^\star \mid \exists m \mathbb{N}\, \exists y_1, \ldots, y_m \in A : x = y_1 y_2 \ldots y_m\}$$

In other words, $A^\star$ consists of all finite concatenations of strings in A.

Example: If $A = \{00, 1\}$, then

$$A^\star = \{\varepsilon, 1, 00, 11, 001, 100, 111, 0000, 0011, 1001, 1100, \ldots\}.$$

(10 points) Prove the following: If $A \in$ NL, then $A^\star \in$ NL.

To do this, sketch a non-deterministic logarithmic space-bounded Turing machine that accepts $A^\star$. Give reasons why your Turing machine correctly accepts $A^\star$ and why it is logarithmic space-bounded.

4. NP and co-NP

Let

$$\text{MAJORITYSAT} = \{F \mid \text{ Boolean formula } F \text{ is satisfied by more than half of all assignments}\}.$$

This means that a Boolean formula F on n variables is contained in MAJORITYSAT if and only if at least $2^{n-1} + 1$ of the possible assignments satisfy the formula.
Let

$$\text{TAUTOLOGY} = \{F \mid \text{ Boolean formula } F \text{ is satisfied by all assignments}\}.$$

(a) Let F be a Boolean formula on n variables, and let y be a Boolean variable that does not appear in F.

(4 points) Prove the following:

$$F \in \text{SAT} \iff F \vee y \in \text{MAJORITYSAT}.$$

(b) Let M_m be the following Boolean formula over variables $y_1, \ldots, y_m$:

$$M_m = y_1 \vee \left(\bigwedge_{i=1}^{m} \overline{y_i} \right).$$

This means that

$$M_m = \begin{cases} 1 & \text{if } y_1 = 1, \\ 1 & \text{if } y_1 = y_2 = \ldots = y_m = 0, \text{ and} \\ 0 & \text{in all other cases.} \end{cases}$$

Note that M_m has exactly $2^{m-1}+1$ satisfying assignments, which is just one more than half of all possible assignments to its variables. (You do not have to prove this.)

Let F be a Boolean formula over n variables, and let $y_1, y_2, \ldots, y_{n+1}$ be Boolean variables that do not appear in F.

(6 points) Prove the following:

$$F \in \text{TAUTOLOGY} \iff F \wedge M_{n+1} \in \text{MAJORITYSAT}.$$

(c) *(4 points)* Taking into account that SAT is NP-complete and that TAUTOLOGY is co-NP-complete, discuss whether MAJORITYSAT is in NP or not.

5. Questions

Are the following statements true or false? Justify your answers.

(a) *(3 points)* If $\mathsf{NP} \neq \text{co-}\mathsf{NP}$, then $\mathsf{P} \neq \mathsf{PSPACE}$.

(b) *(3 points)* If $\mathsf{NP} = \mathsf{P}$, then there exists a constant c such that we have $\mathsf{NP} \subseteq \mathsf{DTime}(O(n^c))$.

(c) *(5 points)* The following statement holds for all sets $A, B, C \subseteq \mathbb{N}$ with $A \subseteq B \subseteq C$: If $A \notin \mathsf{REC}$ and $C \notin \mathsf{REC}$, then $B \notin \mathsf{REC}$.