# Test 2 Resit: Signals and Transforms

MOD-03-AM

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DATE/TIME: May 29th 2026, 13:45 – 15:15

- Closed book, no-calculator test, switched-off mobile phones.
- Grade: $\dfrac{\text{Obtained score}}{\text{Total points}} \times 9 + 1$ (rounded off to one decimal place)
- all answers require rigorous argumentation based on the knowledge from the course.

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**Question 1** (3 points). Let $f\colon \mathbb{R} \to \mathbb{C}$ be an $\mathrm{L}^2$ function such that $g\colon \mathbb{R} \to \mathbb{C}$ defined by $g(t) := tf(2(t-3))$ is also $\mathrm{L}^2$. Express the Fourier transform of $g$ in terms of the Fourier transform of $f$.

**Question 2** (3 points). Show the product rule for the distributional derivative: Let $\psi \in \mathrm{C}^\infty(\mathbb{R})$ and $\Lambda \in \mathcal{D}'(\mathbb{R})$. Then

$$(\psi\Lambda)' = \psi'\Lambda + \psi\Lambda'.$$

**Question 3** (3 points). Let $f, g\colon \mathbb{R} \to \mathbb{C}$ be such that their convolution $f * g$ exists, and $\operatorname{supp} f \subseteq [a, b]$ and $\operatorname{supp} g \subseteq [c, d]$. Show that then

$$\operatorname{supp}(f * g) \subseteq [a + c, b + d].$$

**Question 4** (3 points). Calculate the Fourier transform of $f$ given by $f(t) = \mathbb{1}_{[0,1]}(t) \cdot t$ for $a, b \in \mathbb{R}$ such that $a < b$.

**Question 5** (6 points). Regard the following differential equation

$$y''(t) - 6y'(t) - 7y(t) = u'(t) + u(t) \qquad t > 0,$$

with initial conditions $y(0^-) = 0$, $y'(0^-) = 1$ and input $u(t) = \mathrm{e}^{-2t}$. Find $y(t)$ for $t > 0$.

## Tables for the Fourier transform

| | $f(t)$ | $\hat{f}(\omega)$ |
|---|---|---|
| Time shift | $f(t-\tau)$ | $\mathrm{e}^{-\mathrm{i}\omega\tau}\hat{f}(\omega)$ |
| Time scaling | $f(at)$ | $\frac{1}{\lvert a\rvert}\hat{f}(\frac{\omega}{a})$ |
| Derivative (time) | $\frac{\mathrm{d}}{\mathrm{d}t}f(t)$ | $\mathrm{i}\omega\hat{f}(\omega)$ |
| Frequency shift | $\mathrm{e}^{\mathrm{i}\omega_0 t}f(t)$ | $\hat{f}(\omega-\omega_0)$ |
| Derivative (frequency) | $-\mathrm{i}tf(t)$ | $\frac{\mathrm{d}}{\mathrm{d}\omega}\hat{f}(\omega)$ |
| Almost involution | $\hat{f}(t)$ | $f(-\omega)$ |

| $f(t)$ | $\hat{f}(\omega)$ | condition |
|---|---|---|
| $\sqrt{2\pi}\,\mathrm{rect}_a(t)$ | $a\,\mathrm{sinc}(\frac{a\omega}{2})$ | $a>0$ |
| $\sqrt{2\pi}\,\mathrm{trian}_a(t)$ | $a\,\mathrm{sinc}^2(\frac{a\omega}{2})$ | $a>0$ |
| $\mathrm{e}^{-a\lvert t\rvert}$ | $\dfrac{1}{\sqrt{2\pi}}\dfrac{2a}{a^2+\omega^2}$ | $\mathrm{Re}(a)>0$ |
| $\mathbb{1}_{\mathbb{R}_+}(t)\frac{t^n}{n!}\mathrm{e}^{-at}$ | $\dfrac{1}{\sqrt{2\pi}}\dfrac{1}{(a+\mathrm{i}\omega)^{n+1}}$ | $\mathrm{Re}(a)>0$ |
| $\mathbb{1}_{\mathbb{R}_-}(t)\frac{t^n}{n!}\mathrm{e}^{-at}$ | $\dfrac{1}{\sqrt{2\pi}}\dfrac{1}{(a+\mathrm{i}\omega)^{n+1}}$ | $\mathrm{Re}(a)<0$ |
| $\delta_{t_0}$ | $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}\omega t_0}$ | $t_0\in\mathbb{R}$ |

## Tables for the Laplace transform

| | $f(t)$ | $\mathcal{L}(f)(s)$ |
|---|---|---|
| Time shift | $f(t-\tau)\mathbb{1}_{\mathbb{R}_+}(t-\tau)$ | $\mathrm{e}^{-s\tau}\mathcal{L}(f)(s)\quad(\tau>0)$ |
| Time scaling | $f(at)$ | $\frac{1}{\lvert a\rvert}\mathcal{L}(f)(\frac{s}{a})$ |
| Derivative ($t$) | $\frac{\mathrm{d}}{\mathrm{d}t}f(t)$ | $s\mathcal{L}(f)(s)-f(0^-)$ |
| Shift in $s$ | $\mathrm{e}^{s_0 t}f(t)$ | $\mathcal{L}(f)(s-s_0)$ |
| Derivative ($s$) | $-tf(t)$ | $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}(f)(s)$ |
| Convolution | $(f*g)(t)$ | $\mathcal{L}(f)(s)\cdot\mathcal{L}(g)(s)$ |

| $f(t)$ | $\mathcal{L}(f)(s)$ |
|---|---|
| $\frac{t^n}{n!}\mathrm{e}^{at}$ | $\dfrac{1}{(s-a)^{n+1}}$ |
| $\cos(bt)$ | $\dfrac{s}{s^2+b^2}$ |
| $\sin(bt)$ | $\dfrac{b}{s^2+b^2}$ |
| $\mathrm{e}^{at}\cos(bt)$ | $\dfrac{s-a}{(s-a)^2+b^2}$ |
| $\delta_{t_0}$ | $\mathrm{e}^{-st_0}$ |