Test Linear Structures 2.
Applied Mathematics, 2025-1B: Structures and Systems

This exam consists of 10 problems:
Final Answer: 8 problems
Open Questions: 2 problems.

Final Answer
For the statements, you choose one of three options: true, false, or no answer.
For each correct True or False you will receive (partial) points.
One incorrect True or False results in zero points for that entire question.
Please write all answers to final answer questions on a **single** page.
I advise you to write full words 'True' or 'False':
If your single 'T' resembles an 'F' or vice versa, this could cost points!
Total score for Final Answer: 40 points.
Required score: 20 points.

Open Questions
Write the solutions following two of the four steps.
Step 3. Execute the plan.
Step 4. Analyze your solution and the answer.
Think, for example, of the following questions:
– Can you interpret the solution and/or the result intuitively?
– Does the solution/result make sense in special cases?
– Which role each condition played in the solution?
– Could you relax some assumptions of the problem?
– Is the problem related to other problems or results?
Total score for Open Questions: 40 points.
Step 3: 80%, Step 4: 20%.
Required score: 20 points.
Grade: 1+9(number of points)/80.

A (graphical) calculator is not needed and is **not allowed** at the exam.

PART 1: Multiple choice and Final answer questions

1. [5pt] Consider $P_{10}(\mathbb{R})$, the vector space of polynomials of degree at most 10 over the field $\mathbb{R}$. Note that $\dim(P_{10}(\mathbb{R})) = 11$.
Consider the operator T defined as $T(f) = -f$ on $P_{10}(\mathbb{R})$. Determine the characteristic polynomial of T.

2. [5pt] Give an example of a 2×2 matrix with real coefficients that is not diagonalizable over $\mathbb{C}$.
(that is, the field is $\mathbb{C}$, but the coefficients in your example must be real numbers.)

3. [5pt] Consider a linear operator T on a vector space V. Let $\boldsymbol{v} \in V$. For each of the following statements, indicate whether the statement is true (T) or false (F) or give no answer (N).

   (a) If $\boldsymbol{v}$ is an eigenvector of T, then span($\{\boldsymbol{v}\}$) is a T-invariant subspace of V.

   (b) If $\boldsymbol{v}$ is an eigenvector of T, then span($\{\boldsymbol{v}, T(\boldsymbol{v})\}$) is a T-invariant subspace of V.

   (c) If $T(\boldsymbol{v})$ is an eigenvector of T, then span($\{\boldsymbol{v}, T(\boldsymbol{v})\}$) is a T-invariant subspace of V.

   (d) If $\boldsymbol{v}$ is an eigenvector of T, then span($\{\boldsymbol{v} + T(\boldsymbol{v})\}$) is a T-invariant subspace of V.

4. [5pt] Consider an inner product space space V with $\boldsymbol{u}, \boldsymbol{w} \in V$.
Let β be a basis for V.
Let γ be an orthonormal basis for V.
For each of the following statements, indicate whether the statement is true (T) or false (F) or give no answer (N).

   (a) For all $\boldsymbol{v}$ in V: If $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{w}, \boldsymbol{v} \rangle$, then $\langle \boldsymbol{u} - \boldsymbol{w}, \boldsymbol{v} \rangle = 0$.

   (b) If $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{w}, \boldsymbol{v} \rangle$ for all $\boldsymbol{v}$ in V, then $\boldsymbol{u} = \boldsymbol{w}$.

   (c) If $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{w}, \boldsymbol{v} \rangle$ for all $\boldsymbol{v}$ in β, then $\boldsymbol{u} = \boldsymbol{w}$.

   (d) If $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{w}, \boldsymbol{v} \rangle$ for all $\boldsymbol{v}$ in γ, then $\boldsymbol{u} = \boldsymbol{w}$.

5. [5pt] Consider $\mathbb{R}^3$ with inner product $\langle .,. \rangle$. Note that $\langle .,. \rangle$ is not necessarily the standard inner product.
Let W be a subspace of $V = \mathbb{R}^3$

Let $v = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$. Let $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ be the orthogonal projection of v on W.

Determine a non-zero vector that is orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

6. [5pt] Consider $\mathbb{R}^2$ with inner product $\left\langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle = ac + \frac{1}{2}(ad + bc) + bd.$

Let $A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$.

Determine $(L_A)^* \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$

Hint: It can help to consider $\left\langle L_A \left(\begin{bmatrix} a \\ b \end{bmatrix} \right), \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$.

7. [5pt] Consider a normal operator T with eigenvectors v_1, v_2. For each of the following statements, indicate whether the statement is true (T) or false (F) or give no answer (N).

(a) If v_1 is in the same eigenspace of T as v_2, then v_1 is orthogonal to v_2.
(b) If v_1 is in the same eigenspace of T as v_2, then v_1 is **not** orthogonal to v_2.
(c) If v_1 is in a different eigenspace of T than v_2, then v_1 is orthogonal to v_2.
(d) If v_1 is in a different eigenspace of T than v_2, then v_1 is **not** orthogonal to v_2.

8. [5pt] For each of the following statements, indicate whether the statement is true (T) or false (F) or give no answer (N).

(a) Every orthogonal operator on a real finite-dimensional vector space is diagonalizable.

(b) Every unitary operator on a complex finite-dimensional vector space is diagonalizable.

(c) Every diagonalizable operator on a real finite-dimensional vector space is orthogonal.

(d) Every diagonalizable operator on a complex finite-dimensional vector space is unitary.

PART 2: Open questions.

9. [20pt] Consider a finite-dimensional real vector space V. Consider a **diaganolizable** linear operator $T : V \to V$ with characteristic polynomial $f(t)$.
Prove that $f(T) = T_0$ without using the Cayley-Hamilton Theorem.
In this statement, T_0 denotes the zero-operator, multiplication in $f(t)$ denotes function composition in $f(T)$, and field elements c in $f(t)$ denote operators cI in $f(T)$.

10. [20pt] Consider a finite-dimensional real inner product space V.
Consider a self-adjoint linear operator T on V of which all eigenvalues are non-negative.
Prove that there exists an operator $U : V \to V$ for which $U^*U = T$.
Hint: you can define U by describing how it acts on a basis for V.