LINEAR STRUCTURES

Solution key for the exam of

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P1.

P2. (a) $A \oplus B = A\backslash B \cup B\backslash A = B\backslash A \cup A\backslash B = B \oplus A$

(b) For all $A \in \mathcal{P}(S)$ one has $A \oplus \varnothing = A\backslash\varnothing \cup \varnothing\backslash A = A$. Thus, $0_{\mathcal{P}(S)} = \varnothing$.

(c) For all $A \in \mathcal{P}(S)$ one has $A \oplus A = A\backslash A \cup A\backslash A = \varnothing$. Thus $(-A) = A$.

(d) For all $\lambda, \mu \in \mathbb{Z}/_{\sim_2}$, $A \in \mathcal{P}(S)$: $(\lambda \cdot \mu) \odot A = \begin{Bmatrix} A & \text{if } \lambda = \mu = [1] \\ \varnothing & \text{otherwise} \end{Bmatrix} = \lambda \odot (\mu \odot A)$.

(e) Linearly independent: $\bigoplus_{m=1}^{n} \lambda^m \odot e_m = \varnothing$ implies $\lambda^1 = \lambda^2 = \cdots = \lambda^d = [0]$,
Generating: $A = \bigcup\{\{s_m\} \mid s_m \in A\} = \bigoplus_{m=1}^{n} A^m \odot e_m$, with $A^m = [1]$ if $s_m \in A$ and $A^m = [0]$ otherwise.

(f) $\epsilon^a(A) := \begin{Bmatrix} [1] & \text{if } s_a \in A \\ [0] & \text{if } s_a \notin A \end{Bmatrix}$, whence $\epsilon^a(e_b) = \epsilon^a(\{s_b\}) = \begin{Bmatrix} [1] & \text{if } a = b \\ [0] & \text{if } a \neq b \end{Bmatrix} = \delta^a_b$.

P3. (a) (i) Known that $0_{V^*} = (v \mapsto 0)$, thus $0_{V^*}(s) = 0$ for all $s \in S \subseteq V$ and thus $0_{V^*} \in \operatorname{ann} S$.
(ii) Suppose $\alpha, \beta \in \operatorname{ann} S$, then for all $s \in S$ both $\alpha(s) = 0$ and $\beta(s) = 0$, so that $(\alpha \boxplus \beta)(s) = \alpha(s) + \beta(s) = 0$ for all $s \in S$. Hence $\alpha \boxplus \beta \in \operatorname{ann} S$.
(iii) Suppose $\alpha \in \operatorname{ann} S$, then for all $s \in S$ one has $\alpha(s) = 0$, so that for any $\lambda \in F$ one has $(\lambda \boxdot \alpha)(s) = \lambda \cdot \alpha(s) = 0$ for all $s \in S$, whence $\lambda \boxdot \alpha \in \operatorname{ann} S$.

(b) Suppose $\alpha \in \operatorname{ann} S$, then for all $s \in S$ one has $\alpha(s) = 0$. But then also for all $s \in T \subseteq S$ one has $\alpha(s) = 0$. Thus $\alpha \in \operatorname{ann} T$.

(c) For $T = \{0_V\}$ and $S = V$, which satisfy $T \subseteq S \subseteq V$, one has $\operatorname{ann} S = \{0_{V*}\}$ and $\operatorname{ann} T = V^*$. Thus $\operatorname{ann} S \neq \operatorname{ann} T$ for any non-trivial V, i.e., unless $V = \{0_V\}$ and thus $V^* = \{0_{V*}\}$.

(d) Suppose $\sigma \in \operatorname{im} \varphi^*$, then there exists a $\tau \in V^*$ such that $\varphi^*(\tau) = \sigma$, which means that $\sigma = \tau \circ \varphi$. Now suppose $v \in \ker \varphi$. Thus $\sigma(v) = (\tau \circ \varphi)(v) = \tau(\varphi(v)) = \tau(0_V) = 0_F$ for any $v \in \ker \varphi$, which means that $\sigma \in \operatorname{ann} \ker \varphi$.

P4. (a) Since $e_1, \ldots, e_d$ is a basis and thus a generating set for the vector space, such $v^1, \ldots, v^d$ exist. Now suppose there are $v^1, \ldots, v^d$ and $\tilde{v}^1, \ldots, \tilde{v}^d$ such that $v = v^m \odot e_m$ and $v = \tilde{v}^m \odot e_m$. Then $0 = v - v = (v^m - \tilde{v}^m) \odot e_m$. But since $e_1, \ldots, e_d$ is a basis and hence linearly independent, it follows that $v^m - \tilde{v}^m = 0_F$ for $m = 1, \ldots, d$. Hence the coefficients v^m are unique.

(b) For any $a = 1, \ldots, d$, one has $\epsilon^a(v) = \epsilon(v^m \odot e_m) = v^m \epsilon^a(e_m) = v^m \delta^a_m = v^a$.

(c) Components of covector $\sigma_a := \sigma(e_a)$, components of endo $\varphi^a{}_b := \epsilon^a(\varphi(e_b))$, components of $(1,2)$-tensor $T^a{}_{bc} := T(\epsilon^a, e_b, e_c)$.

(d) $T(\sigma, v, w) = T(\sigma_a \boxdot \epsilon^a, v^b \odot e_b, w^c \odot e_c) = \sigma_a v^b w^c T(\epsilon^a, e_b, e_c) = T^a{}_{bc}\sigma_a v^b w^c$.

(e) $(\alpha \circ \beta)^a{}_b = \epsilon^a((\alpha \circ \beta)(e_b)) = \epsilon^a(\alpha(\beta(e_b))) = \epsilon^a(\alpha(\epsilon^m(\beta(e_b)) \odot e_m)) = \epsilon^m(\beta(e_b))\epsilon^a(\alpha(e_m)) = \alpha^a{}_m \beta^m{}_b$.

P5. Reduce the augmented matrix

$$\left[\begin{array}{cccc|c} 2 & 9 & -7 & 35 & 2\kappa \\ -1 & 16 & -17 & 44 & 3\kappa \\ 0 & 5 & -5 & 15 & \kappa \\ 1 & 3 & -2-\lambda & 12 & 0 \end{array}\right]$$

for the problem to the (non-unique) REF

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 4 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & \lambda & 1 & \kappa \\ 0 & 0 & 0 & 0 & \kappa \end{array}\right]$$

whose solution forces one to distinguish the cases

(a) $\kappa \neq 0$:

No solutions. Thus $S(\lambda, \kappa) = \varnothing$ for all $\kappa \in \mathbb{Q}^*$.

(b) $\kappa = 0$ and $\lambda = 0$:

Backwards substitution yields the equations $v^4 = 0$, $v^2 - v^3 = 0$, $v^1 + v^3 = 0$.

Parametrising $v_3 = s$, one obtains $v^1 = -s$, $v^2 = s$, $v^3 = s$, $v^4 = 0$.

Thus $S(0,0) = \{s \odot (-e_1 \oplus e_2 \oplus e_3) \mid s \in \mathbb{Q}\}$.

(c) $\kappa = 0$ and $\lambda \neq 0$:

Backwards substitution yields the equations $\lambda v_3 + v_4 = 0$, $v_2 - v_3 + 3v_4 = 0$, $v_1 + v_3 + 4v_4 = 0$.

Parametrising $v_4 = s$, one obtains $v_3 = -\frac{s}{\lambda}$, $v_2 = -\frac{s}{\lambda} - 3s$, $v_1 = \frac{s}{\lambda} - 4s$.

Thus $S(\lambda, 0) = \{(\frac{s}{\lambda} - 4s) \odot e_1 + (-\frac{s}{\lambda} - 3s) \odot e_2 \oplus (-\frac{s}{\lambda}) \odot e_3 + s \odot e_4 \mid s \in \mathbb{Q}\}$ for $\lambda \in \mathbb{Q}^*$.

Note that the above solutions merely indicate key steps

to be reached in any fully fleshed out solution.

Solutions are sketched this way in order to allow for easy inspection

and comparison to one's own solutions offered in the exam.