Linear structures 1

Fields

⇒ **Definition**: A field is a triple $(F, +, \cdot)$ where F is a set and $+ := F \times F \to F$

$\cdot := F \times F \to F$

such that $c^{+}a^{+}n^{+}i^{+}\ c^{\cdot}a^{\cdot}n^{\cdot}i^{\cdot}\ D^{+,\cdot}$

$C^{+}: \forall a,b \in F: a+b = b+a$

$A^{+}: \forall a,b,c \in F: (a+b)+c = a+(b+c)$

$N^{+}: \exists 0 \in F: \forall a \in F: a+0 = a$

$I^{+}: \forall a \in F: \exists (-a): a+(-a) = 0$

$C^{\cdot}: \forall a,b \in F: a \cdot b = b \cdot a$

$A^{\cdot}: \forall a,b,c \in F: (a \cdot b) \cdot c = a \cdot (b \cdot c)$

$N^{\cdot}: \exists 1 \in F^{*}: \forall a \in F: a \cdot 1 = a$

$I^{\cdot}: \forall a \in F^{*}: \exists a^{-1}: a \cdot a^{-1} = 1$

$\Big\}\ F^{*} = F \setminus \{0\}$

$D^{+,\cdot}: \forall a,b,c \in F: a(b+c) = a \cdot b + a \cdot c$

2. Property of fields

a) Neutral element of '+' is unique

↳ proof: Suppose 0 and $\tilde{0}$ are both add. neutral:

$$\overset{I^{+}, I^{+}}{\Rightarrow} \forall a \in F: a+0 = a,\ a+\tilde{0} = a$$

But then:

$$\tilde{0} \underset{\uparrow N^{+},\text{ass}}{=} \tilde{0} + 0 \underset{\uparrow C^{+}}{=} 0 + \tilde{0} \underset{\uparrow N^{+},\text{ass}}{=} 0$$

b) For every $a \in F$ there is a unique $-a \in F$

↳ proof: Suppose $(-a)$ and $(\sim a)$ are both addition inverses of a

$$\overset{I^{+}}{\Updownarrow}\quad a+(\sim a) = 0,\ a+(-a) = 0$$

$$(\sim a) \overset{N^{+}}{=} (\sim a) + 0 \overset{\text{ass.}}{=} (\sim a) + (a+(-a)) \overset{A^{+}}{=} ((\sim a)+a)+(-a) \overset{C^{+}}{=}$$

$$(a+(\sim a)) + (-a) \overset{\text{ass.}}{=} 0+(-a) \overset{C^{+}}{=} (-a)+0 \overset{N^{+}}{=} (-a)$$

c) Neutral element of '·' is unique

↳ proof: Suppose 1 and $\tilde{1}$ are both multp. neutral:

$$\overset{I^{\cdot}, I^{\cdot}}{\Rightarrow} \forall a \in F: a \cdot 1 = a,\ a \cdot \tilde{1} = a$$

But then

$$\tilde{1} \underset{\uparrow N^{\cdot},\text{ass}}{=} \tilde{1} \cdot 1 \underset{\uparrow C^{+}}{=} 1 \cdot \tilde{1} \underset{\uparrow N^{\cdot},\text{ass}}{=} 1$$

d) For every $a \in F$ there is a unique $a^{-1} \in F$:

↳proof: Suppose a^{-1} and $\tilde{a}^{-1}$ are both multiplicative inverses of a $\qquad a \cdot a^{-1} = 1,\ \overset{\updownarrow I\cdot}{a} \cdot \tilde{a}^{-1} = 1$

$\tilde{a}^{-1} \overset{\downarrow N\cdot}{=} \tilde{a}^{-1} \cdot 1 \overset{\downarrow ass.}{=} \tilde{a}^{-1} \cdot (a \cdot a^{-1}) \overset{\downarrow A\cdot}{=} (\tilde{a}^{-1} \cdot a)\, a^{-1} \overset{\downarrow C+}{=} (a \cdot \tilde{a}^{-1}) \cdot a^{-1} \overset{\swarrow ass}{=}$

$1 \cdot a^{-1} \overset{\downarrow C+}{=} a^{-1} \cdot 1 \overset{\downarrow N\cdot}{=} a^{-1}$

e) $\forall a \in F : 0 \cdot a = 0$

↳proof: $a + 0 \cdot a \overset{\downarrow N\cdot}{=} 1 \cdot a + 0 \cdot a \overset{\downarrow c,c\cdot}{=} a \cdot 1 + a \cdot 0 \overset{\downarrow D+,\cdot}{=} a(1+0) \overset{\swarrow N+}{=}$

$a \cdot 1 \overset{\downarrow C+}{=} 1 \cdot a \overset{\downarrow N\cdot}{=} a$

$\overset{N+}{\Rightarrow}$ Thus $0 \cdot a$ is a additively neutral element

$\overset{a)}{\Rightarrow} 0 \cdot a = 0$

f) $\forall a \in F : (-1) \cdot a = -a$

↳proof:

g) $\forall a, b \in F : a \cdot b = 0 \Rightarrow a = 0 \vee b = 0$

↳proof:

3. Examples

a) $(\mathbb{Q}, \boxed{+}, \boxed{\cdot})$ from Fom

b) $(\mathbb{R}, +, \cdot)$ as axiomatic introduced in analysis 1

c) $(\mathbb{C}, \oplus, \odot)$ "field of complex numbers"

d) $(\mathbb{Z}/_{\sim p}, \circledast, \circledcirc)$ field

$\overset{\shortparallel}{GF(p)}$ ↖ prime number

e) generalisation of $(GF(p))$ (galios fields) is $GF(p^k)$ $k \in \mathbb{N}^*$ (not in exam)

4. Special case: field of complex numbers $(\mathbb{C},\oplus,\odot)$

=> **Definition:** $\mathbb{C} = \mathbb{R}\times\mathbb{R}$ $\quad (0_{\mathbb{C}} = (0,0),\ 1_{\mathbb{C}} = (1,0))$

=> **Definition:** $\oplus : \mathbb{C}\times\mathbb{C}\to\mathbb{C}$

$$(a,b)\oplus(c,d) := (a+c,\ b+d)$$

=> **Definition:** $\odot : \mathbb{C}\times\mathbb{C}\to\mathbb{C}$

$$(a,b)\odot(c,d) := (ac-bd,\ ad+bc)$$

Now we can check whether $(\underbrace{\mathbb{C},\overbrace{\oplus,\odot}}_{D^{+,\cdot}})$ is a field

↳proof:

=> **Definition:** $\mathrm{com}: \mathbb{R}\hookrightarrow\mathbb{C}$

$$a \mapsto (a,0)$$

=> Theorems.

a) com is injective

↳proof: let

$$\begin{array}{ccc} \mathrm{com}(a) & = & \mathrm{com}(b) \\ \| & & \| \\ (a,0) & & (b,0) \\ \| & & \| \\ \{\{a\},\{a,0\}\} & & \{\{b\},\{b,0\}\} \\ & \Downarrow \text{'='} & \\ & a=b & \qquad \text{QED} \end{array}$$

b) $\mathrm{com}(a\overset{\mathbb{R}}{+}b) = \mathrm{com}(a)\oplus\mathrm{com}(b)$

$\mathrm{com}(a\overset{\mathbb{R}}{\cdot}b) = \mathrm{com}(a)\cdot\mathrm{com}(b)$

$\mathrm{Com}(1_{\mathbb{R}}) = 1_{\mathbb{C}}$

↳proof:

=> Traditional notation:

1. $i := (0,1)$
2. $a = \mathrm{Com}(a) = \mathrm{Com}(a) = (a,0)$
3. write $\cdot := \odot$, $+ := \oplus$

Two fun facts using this traditional notation:

a) $\underset{\uparrow \text{ more notation}}{i^2 :=} i \odot i = (0,1) \odot (0,1) \underset{\uparrow \text{ def } \odot}{=} (-1,0) = \mathrm{com}(-1) = \mathrm{com}(-1) = -1$

b) $\forall z \in \overset{\mathbb{R}\times\mathbb{R}}{\mathbb{C}} \;\exists a,b \in \mathbb{R} : z = a \oplus ib$

$\hookrightarrow$ proof: $\exists a,b \in \mathbb{R} : z = (a,b) = (a,0) \oplus \overbrace{(0,1) \cdot (b \cdot 0)}^{(0,b)}$

$= \mathrm{com}(a) \oplus \mathrm{com}(b) \odot i$

$= a \oplus ib = a + b$

5. Finite fields

$\Rightarrow$ **Definition:** A field $(F,+,\cdot)$ is called finite if F is a finite set

$\Rightarrow$ **Definition:** $\underset{\text{order}}{\mathrm{ord}}(F) = \underset{\text{number of elements in } F}{|F|}$

$\Rightarrow$ **Definition:** For a finite field $(F,+,\cdot)$

$\underset{\text{characteristic}}{\mathrm{char}}(F) =$ minimal positive number of times that one must add

$\underbrace{1_F + 1_F + 1_F \ldots + 1_F}_{\mathrm{char}(F) \text{ times}} \overset{!}{=} 0_F$

$\Rightarrow$ **Remark:** char(infinite field) $:= 0$

$\Rightarrow$ Prototypical example of a finite field $\mathbb{Z}$

let p be a prime number

let $\sim_p \subseteq \mathbb{Z} \times \mathbb{Z}$ such that

$S \sim_p b \Leftrightarrow \exists n \in \mathbb{Z} : a - b = n \cdot p$

$(\mathbb{Z}/\sim_p, \oplus, \odot)$ $\oplus : \mathbb{Z}/\sim_p \times \mathbb{Z}/\sim_p \to \mathbb{Z}/\sim_p$

$[a] \oplus [b] := [a \overset{\mathbb{Z}}{+} b]$

$\odot : \mathbb{Z}/\sim_p \times \mathbb{Z}/\sim_p \to \mathbb{Z}\sim_p$

$[a] \odot [b] = [a \overset{\mathbb{Z}}{\cdot} b]$

Well-definedness $\odot$ (needed wenn domain is a quotionset))

$\underbrace{[a]}_{\shortparallel} \odot \underbrace{[b]}_{\shortparallel} = [a \cdot b]$

$[a + \underset{\uparrow \text{ any } n \in \mathbb{Z}}{n} \cdot p]\,[b + m \cdot p]$ (analogously for $\oplus$)

Need: $[a+np] \odot [b+mp] \overset{?}{=} [a] \odot [b]$

$[(a\overset{\mathbb{Z}}{+}np)\cdot(b+mp)] = [a\cdot b + amp + bnp + mnp] =$

$[a\cdot b + (a\cdot m + b\cdot n + m\cdot n)p] = [a\cdot b] = [a] \odot [b]$

↳ Subexample: $\mathbb{Z}/\sim_3 = \{[0],[1],[2]\}$

$\odot$ b\a	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

$(\mathbb{Z}/\sim_3)^*$

↳ Subcounterexample: $(\mathbb{Z}/\sim_4)^*$

$\odot$ b\a	[0]	[1]	[2]	[3]
[0]	~~[0]~~	~~[0]~~	~~[0]~~	~~[0]~~
[1]	~~[0]~~	[1]	[2]	[3]
[2]	~~[0]~~	[2]	[0] ↯	[2]
[3]	~~[0]~~	[3]	[2]	[1]

not a field

p not prime

$\exists m,n \in \mathbb{N} : p = m\cdot n$

$[p] = [0]$ ↯

Vectorspaces over a field

Definition of the key object of study

⇒ **Definition**: Let $(F,+,\cdot)$ be a field. Then a vectorspace over F (F-vectorspace) is a triple $(V,\oplus,\odot)$

{a(2Fc)set} → V

↳ $\oplus : V\times V \to V$ 'addition on V'

↳ $\odot : F\times V \to V$ 'scaling on V'

Such that: CANI ADDU

$C^{\oplus}$ $\forall v,w \in V: v\oplus w = w\oplus v$

$A^{\oplus}$ $\forall v,w,u \in V: (v\oplus w)\oplus u = v\oplus(w\oplus u)$

$N^{\oplus}$ $\exists 0_V \in V : \forall v\in V : v\oplus 0_V = v$

$I^{\oplus}$ $\forall v \in V : \exists (-v) \in V : v + (-v) = 0$

$A^{\cdot,\odot}$ $\forall \lambda,\mu \in F\ \forall v \in V: \lambda\odot(\mu\odot v) = (\lambda\cdot\mu)\odot v$

$D^{\oplus,\odot}$ $\forall \lambda \in F\ \forall v,w \in V: \lambda\odot(v\oplus w) = \lambda\odot v\oplus\lambda\odot w$

$D^{+,\oplus,\odot}$ $\forall \lambda\mu \in F\ \forall v \in V: (\lambda+\mu)\odot v = \lambda\odot v \oplus \mu\odot v$

U $\forall v \in V: 1_F \odot v = v$

⇒ Remarks:

1) Obviously, CANI ADDU require extensive neutral relationships between $(F, +, \cdot, V, \oplus, \odot)$, but there are still many different concrete implementations (examples) of this struggle

2) "Jargon": 'let v be a vector'
There is no such thing as 'a vector'. The only legal use of the word vector is this: "let v be an element of the set V that underlies the vectorspace $(V, \oplus, \odot)$ over the field $(F, +, \cdot)$ [That I have in mind and I assume you do too]"
↳ claim: $v = 7$ is a vector $\wedge$ 7 is not a vector
$v =$ ❀ " $\wedge$ ❀ "
$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ " $\wedge$ $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ "
$v = \left(\begin{array}{l} f: \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 \end{array} \right)$ " $\wedge$ $\begin{array}{l} f: \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 \end{array}$ "

3)

position vector	✗
velocity vector	✓
acceleration vector	✓
angular mom vector	✗
wave vector	✗
momentum vector	✗
gradient vector	✗

4) The same structure, but over a ring $(R, +, \cdot)$ instead of a field is called an R-module.

⇒ Theorems:

1) 0_V is unique } proof precisely the
2) $(-v)$ is the unique add. inv of $v \in V$ } same as fields
3) $0_F \odot v = 0_V$ for all $v \in V$ } proof similar to those proofs on
4) $(-1_F) \odot v = (-v)$ } fields, but use sometimes other laws

5) $\lambda \odot 0_v = 0_v$ for all $\lambda \in F$

↳ proof: let $v \in V$. Consider:

case 1: $v + 0_v = v$

$\Rightarrow v \oplus \lambda \odot 0_v \overset{u}{=} 1_F \odot v \oplus \lambda \cdot 0_v \overset{I^{\cdot}}{=} (\lambda \cdot \lambda^{-1}) \odot v \oplus \lambda \odot 0_v \overset{A^{\cdot,\odot}}{=}$

$\lambda \odot (\lambda^{-1} \odot v) \oplus \lambda \odot 0_v \overset{D^{\oplus\odot}}{=} \lambda \odot (\lambda^{-1} \odot v \oplus 0_v) \overset{N^{\oplus}}{=} \lambda \odot (\lambda^{-1} \odot v) \overset{A^{\oplus,\odot}}{=}$

$(\lambda \cdot \lambda^{-1}) \odot v \overset{I^{\cdot}}{=} 1_F \odot v \overset{u}{=} v$

Case 2: $\lambda = 0 \in F : 0_F \odot 0_v \overset{(3)}{=} 0_v$

$\Rightarrow \lambda \odot 0_v$ is the neutral element of $\oplus$ i.e 0_v QED

=> Remark: note the 'proof technique": prove unique properties of the result

=> Examples:

0) $V := \text{✿}$ field $(F, +, \cdot)$

$\oplus : V \times V \to V$

$\text{✿} \oplus \text{✿} \mapsto \text{✿}$

$\odot : F \times V \to V$

$\lambda \odot \text{✿} \mapsto \text{✿}$

$C^{\oplus}$ $\text{✿} + \text{✿} = \text{✿} + \text{✿}$ ✓

$A^{\oplus}$ $(\text{✿} \oplus \text{✿}) \oplus \text{✿} = \text{✿} \oplus (\text{✿} \oplus \text{✿})$ ✓

$N^{\oplus}$ $0_v := \text{✿}$ ✓

$I^{\oplus}$ $(-\text{✿}) := \text{✿}$ ✓

$A^{\cdot,\odot}$ $\lambda \odot (\mu \odot \text{✿}) = (\lambda \cdot \mu) \odot \text{✿}$ ✓

$D^{\oplus,\odot}$ $\lambda \odot (\text{✿} \oplus \text{✿}) = \lambda \odot \text{✿} \oplus \lambda \odot \text{✿}$ ✓

$D^{+,\oplus,\odot}$ $(\lambda + \mu) \odot \text{✿} = \lambda \odot \text{✿} \oplus \mu \odot \text{✿}$ ✓

A $1_F \odot \text{✿} = \text{✿}$ ✓

a) $V := F$

$\oplus : V \times V \to V$

$v \oplus v := v + v$

$\odot : F \times V \to V$

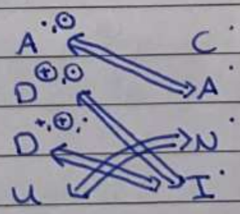
$\lambda \odot v := \lambda \cdot v$

$C^{\oplus} \Leftrightarrow C^{+}$

$A^{\oplus} \Leftrightarrow A^{+}$

$N^{\oplus} \Leftrightarrow N^{+}$

$I^{\oplus} \Leftrightarrow I^{+}$



"F is an F vectorspace if you make these choices."

↳ Subexample: $\mathbb{R}$ is an $\mathbb{R}$-v.s.

$\mathbb{C}$ is an $\mathbb{C}$-v.s.

b) "$\mathbb{C}$ is an $\mathbb{R}$-V.S"

To see this:

Choose $V := \mathbb{C}$

$$\oplus : V \times V \to V$$

$$v \oplus w := v +_{\mathbb{C}} w$$

$$\odot : \overset{\mathbb{R}}{F} \times V \to V \qquad \overset{\in\mathbb{R}}{\downarrow} \quad \overset{\overset{\in\mathbb{R}\ \in\mathbb{R}}{\downarrow\ \ \downarrow}}{(a,b)}$$

$$\lambda \odot v := \mathrm{com}(\lambda) \cdot_{\mathbb{C}} v \quad (:= \lambda \cdot v)$$

$C^{\oplus}$ $\forall v,w \in \mathbb{C} : v +_{\mathbb{C}} w = w +_{\mathbb{C}} v$

$A^{\oplus}$ $\forall v,w,u \in \mathbb{C} : (v +_{\mathbb{C}} w) +_{\mathbb{C}} u = v +_{\mathbb{C}} (w +_{\mathbb{C}} u)$

$N^{\oplus}$ $\exists 0_V \in \mathbb{C} : \forall v \in \mathbb{C} : v +_{\mathbb{C}} 0_V = v$

$I^{\oplus}$ $\forall v \in \mathbb{C} : \exists (-v) \in \mathbb{C} : v +_{\mathbb{C}} (-v) = 0$

$A^{\cdot,\odot}$ $\forall \lambda,\mu \in \mathbb{R}\ \forall v \in \mathbb{C} : \mathrm{com}(\lambda) \cdot_{\mathbb{C}} (\mathrm{com}(\mu) \cdot_{\mathbb{C}} v) = \mathrm{com}(\lambda \cdot_{\mathbb{R}} c) \cdot_{\mathbb{C}} v$

$D^{\oplus,\odot}$ $\forall \lambda \in \mathbb{R}\ \forall v,w \in \mathbb{C} : \mathrm{com}(\lambda) \cdot_{\mathbb{C}} (v \cdot_{\mathbb{C}} w) = \mathrm{com}(\lambda) \cdot_{\mathbb{C}} v + \mathrm{com}(\lambda) \cdot_{\mathbb{C}} w$

$D^{+,\oplus,\odot}$ $\forall \lambda,\mu \in \mathbb{R}\ \forall v \in \mathbb{C} : \mathrm{com}(\lambda +_{\mathbb{R}} \mu) \cdot_{\mathbb{C}} v = \mathrm{com}(\lambda) \cdot_{\mathbb{C}} v +_{\mathbb{C}} \mathrm{com}(\mu) \cdot_{\mathbb{C}} v$

of zander com

U $\forall v \in \mathbb{C} : 1_{\mathbb{R}} \odot v = v$

c) def. $\overset{n \in \mathbb{N}\ \searrow}{P_{\mathbb{R}}^{n}} := \{\underbrace{p : \mathbb{R} \to \mathbb{R}}_{P \subseteq R \times R}, p(x) := \sum_{i=0}^{n} \lambda_i \cdot x^i \mid \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}\}$ is a ZFC set

$$\widehat{\Updownarrow}$$

$$\underbrace{p \in \mathcal{P}\underbrace{(\underbrace{\mathbb{R}}_{ZFC} \times \mathbb{R})}_{ZFC}}_{ZFC}$$

def. $V := P_{\mathbb{R}}^{n}$ over $(\mathbb{R}, +, \cdot)$

$$\oplus : V \times V \to V$$

$\forall x \in \mathbb{R} : (p \oplus q)(x) := p(x) +_{\mathbb{R}} q(x)$ "pointwise def."

$$\odot : \mathbb{R} \times V \to V$$

$$(\lambda \odot p)(x) := \lambda \cdot_{\mathbb{R}} \overbrace{p(x)}^{\in\mathbb{R}}$$

$C^{\oplus}$

$A^{\oplus}$

$N^{\oplus}$

$I^{\oplus}$

$A^{\cdot,\odot}$

$D^{\oplus,\odot}$

$D^{+,\oplus,\odot}$

U

d) $V := F^n \underset{\text{notation}}{:=} \underbrace{F \times F \times \ldots \times F}_{n \text{ cart. factors}} = \{(f_1, \ldots, f_n) \in \ldots \mid f_1, \ldots, f_n \in F\}$

$\oplus : F^n \times F^n \to F^n$

$(f_1, \ldots, f_n) \oplus (g_1, \ldots, g_n) := (f_1 + g_1, \ldots, f_n + f_g)$

$\odot : F \times F^n \to F^n$

$\lambda \odot (f_1, \ldots, f_n) := (\lambda \cdot f_1, \ldots, \lambda \cdot f_n)$

↳proof $(F^n, \oplus, \odot)$ is F-v.s. : all properties inherited from $(F, +, \cdot)$

Morphisms

=> **Remark:** In this entire section $(V, \oplus, \odot)$ $(W, \boxplus, \boxdot)$ are vector spaces
(extra structure on top of set structure)

1. Definitions

=> **Definition:** A map $f : V \to W$ is called a **homomorphism** if

$$\left.\begin{aligned} \forall v, \tilde{v} \in V, \forall \lambda \in F : i)\ & f(v \oplus \tilde{v}) = f(v) \boxplus f(v) && \text{"additivety"} \\ ii)\ & f(\lambda \odot v) = \lambda \boxdot f(v) && \text{"scalability"} \end{aligned}\right\} \text{"linearity"}$$

=> **Remark:** other terminology "linear map", "linear transformation"

=> Terminology:

vectorspace monomorphism :: homomorphism + injective

vectorspace epimorphism := homomorphism + surjective

vectorspace isomorphism := homomorphism + bijective

vectorspace endomorphism := $W = V, \boxplus = \oplus, \boxdot = \odot$

vectorspace automorphism := $W = V, \boxplus = \oplus, \boxdot = \odot$ + bijective

2. Examples

(a) $f : \mathbb{R}^2 \to \mathbb{R}^2$ for some $a, b, c, d \in \mathbb{R}$, field $(\mathbb{R}, +, \cdot)$, $(\mathbb{R}^2, \oplus, \odot)$ as in lecture 2

$(x, y) \mapsto (ax + by, cx + dy)$

↳check: $f((x,y) \oplus (n,v)) =$ proof $= f((x,y)) \oplus f((n,v))$ "add"

$f(\lambda \odot (x,y)) \overset{\odot}{=} f((\lambda \cdot x, \lambda \cdot y)) \overset{f}{=} (a \cdot \lambda \cdot x + b \cdot \lambda \cdot y, c \cdot \lambda \cdot x + d \cdot \lambda \cdot y)$

$\overset{A\cdot, C\cdot, A\cdot}{=} (\lambda(ax) + \lambda(by), \lambda(cx) + \lambda(dy))$

$\overset{D^{+,\cdot}}{=} (\lambda(ax + by), \lambda(cx + dy))$

$\overset{\odot}{=} \lambda \odot (ax + by, cx + dy) \overset{f}{=} \lambda \odot f(x,y)$ "scaling"

Thus f is homo

(b) define set $C^{\infty}(\mathbb{R}) := \{ f: \mathbb{R} \to \mathbb{R} \mid f \text{ is arbitrarily often differentiable} \}$

"Set of smooth functions"

$\oplus : C^{\infty}(\mathbb{R}) \times C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R})$

$(f \oplus g)(x) := f(x) +_{\mathbb{R}} g(x)$ {define pointwise}

$\odot : \mathbb{R} \times C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R})$

$(\lambda \odot f)(x) := \lambda \cdot_{\mathbb{R}} f(x)$

↳ Can show: $(C^{\infty}(\mathbb{R}), \oplus, \odot)$ is an $\mathbb{R}$-vector space *

define: $' : C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R})$ $\quad$ $'$ is homo

$f \mapsto f' := f'$ ∘∘∘ {as in analysis}

↳ check: • Sumrule $(f \oplus g)' = f' \oplus g'$ *

• productrule $(\lambda \odot f)' = \lambda \odot f'$

=> Remark: notation ~~$f(x)'$~~ $f'(x)$

(c) define $\mathbb{R}_+ := \{ r \in \mathbb{R} \mid r > 0 \}$

$\oplus : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$

$r \oplus s := r \cdot_{\mathbb{R}} s$

$\odot : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$

$\lambda \odot r = r^{\lambda}$

On P1: $(\mathbb{R}_+^n, \oplus, \odot)$ is an $\mathbb{R}$-vector space,

here: $n = 1 \Rightarrow (\mathbb{R}_+, \oplus, \odot)$ is an $\mathbb{R}$-vector space.

Consider the map $\ln : \overset{\{\oplus,\odot\}}{\mathbb{R}_+} \to \overset{\{\boxplus,\boxdot\}}{\mathbb{R}}$

$r \mapsto \ln(r)$

↳ check: $\ln(r \oplus s) \overset{\oplus}{=} \ln(r \cdot s) \overset{\text{ana}}{=} \ln(r) + \ln(s) \overset{\boxplus}{=} \ln(r) \boxplus \ln(s)$

so, additive

$\ln(r \odot s) \overset{\odot}{=} \ln(r^{\lambda}) \overset{\text{ana}}{=} \lambda \cdot \ln(r) \overset{\boxdot}{=} \lambda \boxdot \ln(r)$ so, scaling

Thus the above map is a homomorphism

3. kernel and image of a homomorphism

=> Definition: the kernel of a homomorphism φ is the set

$\ker \varphi := \{ v \in V \mid \varphi(v) = 0_W \} \subseteq V$

=> Definition: the image of a homo f is the set
$\operatorname{im} f := \{w \in W \mid \exists v \in V : f(v) = w\}$

=> Theorems: "f is homo"

(1) $0_V \in \ker f$
↳ proof: $f(0_V) \overset{\text{before}}{=} f(0_F \odot 0_V) \overset{f\text{homo}}{=} 0_F \boxdot f(0_V) \overset{\text{before}}{=} 0_W \overset{\text{ker}}{\iff} 0_V \in \ker f$

(2) $v, \tilde{v} \in \ker f \Rightarrow (v \oplus \tilde{v}) \in \ker f$
↳ proof: $\ldots \overset{\text{ker}}{\iff} f(v) = 0_W,\ f(\tilde{v}) = 0_W \overset{N^{\boxplus}}{\Rightarrow} f(v) \boxplus f(\tilde{v}) = 0_W$
$\overset{f\text{homo}}{\iff} f(v \oplus \tilde{v}) = 0_W$
$\overset{\substack{\text{def.}\\ \text{ker}}}{\iff} (v \oplus \tilde{v}) \in \ker f$

(3) $v \in \ker f \Rightarrow (\lambda \odot v) \in \ker f$ for any $\lambda \in F$
↳ proof: $\ldots \overset{\text{ker}}{\iff} f(v) = 0_W \overset{\text{def.}0_W}{\Rightarrow} \lambda \boxdot f(v) = 0_W$
$\overset{f\text{homo}}{\iff} f(\lambda \odot v) = 0_W$
$\overset{\substack{\text{def.}\\ \text{ker}}}{\iff} (\lambda \odot v) \in \ker f$

=> Remark: $\ker f$ is not only a subset of f, but even a so called vector subspace of V (→ soon)

=> Theorem: homo $f: V \to W$ is injective
iff $\ker f = \{0_V\}$

↳ proof:
"=>" know $0_V \in \ker f$, now suppose there exists $v \in V : v \in \ker f$
Thus $f(v) \overset{\substack{\text{def.}\\ \text{ker}}}{=} 0_W$ and $f(0_V) = 0_W$
$\Rightarrow f(v) = f(0_V) \Rightarrow v = 0_V$
"<=" know $\ker f = \{0\}$, suppose $f(v) = f(\tilde{v})$
$\Rightarrow f(v) \boxplus (-f(\tilde{v})) = 0_W$
$\overset{f\text{homo}}{=} f(v \oplus (-\tilde{v})) = 0_W$
$\Rightarrow (v \oplus (-\tilde{v})) \in \ker f,\ f = \{0\}$
$\overset{\substack{\text{def.}\\ \text{ker}}}{\iff} v \oplus (-\tilde{v}) = 0_V \iff v = \tilde{v}$
Thus f injective

iff ≙ $\overset{\Leftarrow}{\text{if}}$ and $\overset{\Rightarrow}{\text{only}}$ if ≙ <=>
a if b ≙ a <= b
a i~~f and~~ only if ≙ a => b
f injective $:\iff f(v) = f(\tilde{v}) \Rightarrow v = \tilde{v}$

Supspaces, quotient and FTH

$\Rightarrow$ Remark: in this entire section $(V, \oplus, \odot)(W, \boxplus, \boxdot)$ are $\mathbb{F}$-vector spaces

"So far we created/defined "structure" now we will use those structures (especially vector spaces and homomorphisms) in order to built new structure ("induce" new structure). This is again a recurrent theme in maths."

1. Subspaces

ooo (Roughly: "space = set + structure)

$\Rightarrow$ Definition: A subset $U \subseteq V$ is called a vector/linear subspace of $(V, \oplus, \odot)$ if:

a) $U \neq \emptyset \quad (0 \in U)$

b) $v, \tilde{v} \in U \Rightarrow v \oplus \tilde{v} \in U$

c) $\lambda \in \mathbb{F}, v \in U \Rightarrow \lambda \odot v \in U$

↳ Notation: $U \leq V$

$\Rightarrow$ Corollary: $U \leq V \Rightarrow 0_V \in U$

↳ proof: $U \neq \emptyset \Rightarrow \exists u \in U \leq V$

$$\Rightarrow U \ni \underset{\overset{\uparrow}{(c)}}{0_{\mathbb{F}}} \odot u \underset{\overset{\uparrow}{\text{lecture 2}}}{=} 0_V$$

$\Rightarrow$ Examples:

a) $f: V \to W$ homomorphism

$\ker f \leq V$

b) $\operatorname{im} f \leq W$

c) $\{0_V\} \leq V$

d $V \leq V$

$\Rightarrow$ Theorem: let C be a set that contains as elements only supspaces of V ooo ($\cap C$. $\{u \in V \mid \forall x \in C : u \in X\}$).
Then $\cap C \leq V$

↳ proof: a) $\Rightarrow \forall x \in C : X \neq \emptyset \Rightarrow \forall x \in C : 0_V \in \cap C \Rightarrow \cap C$ satisfies prop. a

b) Suppose $v, \tilde{v} \in \cap C \Rightarrow \forall x \in C : v \in X \wedge \tilde{v} \in X$

$$\underset{\substack{X \leq V \\ \text{by ass}}}{\Rightarrow} \forall x \in C : v \oplus \tilde{v} \in X \underset{\substack{\text{def.} \\ \cap}}{\Rightarrow} v \oplus \tilde{v} \in \cap C$$

c) $\lambda \in F, v \in \bigcap C \Rightarrow \forall x \in C : v \in X$

$\Rightarrow \forall x \in C : \lambda \odot v \in X \overset{\text{def.} \cap}{\Rightarrow} \lambda \odot v \in \bigcap C$

=> **Definition**: let $S \subseteq V$ any subset of V

Then S induces a subspace

$\text{Span}(S) := \bigcap \{ \underset{\underset{U \in \mathcal{P}(V)}{\Downarrow}}{U \leq V} \mid S \subseteq U \}$

"The span of S is the intersection of all subspaces of V that contain S as a subset."

↳ proof: $S \subseteq V \overset{\text{by theo}}{\Rightarrow} \text{span}(S) \leq V$

(S is infinite if $\exists T \subsetneq S$: $T \cong_{\text{set}} S$ (bijective))

=> **Remarks**: • this may well be an infinite set

• Spans will come back in this course

• $\underset{\underset{V}{\cap\!\!|}}{\text{Span}(\emptyset)} = \{0_V\} \leq V$

=> Example: $2\mathbb{N} := \{2n \in \mathbb{N} \mid n \in \mathbb{N}\}$

$f : \mathbb{N} \to 2\mathbb{N}$

$n \mapsto 2n$ $\quad\Rightarrow\quad f \circ f^{-1} = id_{2\mathbb{N}}$

$f^{-1} : 2\mathbb{N} \to \mathbb{N}$ $\quad\quad f^{-1} \circ f = id_{\mathbb{N}}$

$2n \mapsto n$

2. Quotient spaces

=> **Definition**: Let $U \leq V$. Then define $\sim \subseteq V \times V$

through: $v \sim_U \tilde{v} :\Leftrightarrow v - \tilde{v} \in U$

↳ Claim: $\sim_U$ is an equivalent relation

i) reflexibility: $v \sim_U v \Leftrightarrow v - v \in U \Leftrightarrow 0_V \in U$

ii) symmetry: $v \sim_U \tilde{v} \Leftrightarrow v - \tilde{v} \in U \ldots \Leftrightarrow \tilde{v} - v \in U \Leftrightarrow \tilde{v} \sim_U v$

iii) transibility: !

=> **Remarks**:

1) A subspace induces an equivalent relation induces a quotient set $U \rightsquigarrow \sim_U \rightsquigarrow V/\sim_U := \{[v]_{\sim_U} \in \mathcal{P}(V) \mid v \in V\}$

2) notation: $V/U := V/\sim_U$

3) Point 1 in other words: $U \leq V$ induces set V/U

⇒ **Definition:** $U \leq V$, equip V/U with

$$\oplus : V/U \times V/U \to V/U$$
$$[v] \oplus [\tilde{v}] := [v + \tilde{v}]$$
$$\odot : F \times V/U \to V/U$$
$$\lambda \odot [v] := [\lambda \odot v]$$

↳ proof: (well-definedness) !

⇒ Theorem: $U \leq V$, then $(V/U, \oplus, \odot)$ is an F-vector space called the quotient vector space of V with respect to U.

⇒ **Definition:** $\pi : \underset{\oplus\,\odot}{V} \to \underset{\oplus\,\odot}{V/U}$ $\quad v \to [v]_{\sim U}$

Canonical quotient projection

↳ claim: π is homomorphism !

3. The fundamental theorem of an homomorphism

⇒ Theorem: Let $\varphi : V \to W$ be a homomorphism

Then $\ker\varphi \leq V$, $\operatorname{im}\varphi \leq W$

Then $\underset{\oplus\,\odot}{V/\ker\varphi} \cong_{vec} \operatorname{im}\varphi$

thus quotientspace

($\exists\, \varrho : V/\ker\varphi \to \operatorname{im}\varphi$ that is an vector space isomorphism)

↳ proof:

$$\begin{tikzcd}
V \arrow[r, "\varphi"] \arrow[d, "\pi"'] \arrow[dr, dashed, bend right=60, "\varrho\circ\pi"'] & W \\
V/\ker\varphi \arrow[r, "\varrho"] & \operatorname{im}\varphi \arrow[u, hook, "\iota"']
\end{tikzcd}$$

$S \leq W$

Canonical injective map

$\iota : S \to W$

$s \mapsto s$ monomorphism

Construct $\varrho \circ \pi : V \to \operatorname{im}\varphi$

$$v \mapsto \varrho(\pi(v)) = \underbrace{\varrho([v]) := \varphi(v)}_{\text{check well-definedness}} \in \operatorname{im}\varphi \text{ thus } \varphi \text{homo}$$

well-definedness: $\varrho([v+u]) = \varphi(v \oplus u) \overset{\ker\varphi}{\overset{homo}{=}} \varphi(v) \boxplus \varphi(u) = \varphi(v)$

$= \varrho([v])$ ✓

is ϱ isomorphic?

a) ϱ is epimorphic

↳ obviously

b) ϱ is monomorphic

↳ proof: suppose $\varrho([v]) = \varrho([\tilde{v}]) \overset{\varrho}{\Longleftrightarrow} \varphi(v) = \varphi(\tilde{v}) \overset{I^{\oplus}}{\Longleftrightarrow} \varphi(v) - \varphi(\tilde{v}) = 0_W$

$\underset{\varphi\text{homo}}{\Longleftrightarrow} \varphi(v - \tilde{v}) = 0_W \Leftrightarrow v - \tilde{v} \in \ker\varphi \Leftrightarrow v \sim_{\ker\varphi} \tilde{v} \Leftrightarrow [v] = [\tilde{v}]$

(Section 5) Dual of a vectorspace and multilinear maps

↳ throughout the section $(V,+,\cdot)$ $(W,+,\cdot)$ are $\mathbb{F}$-vector spaces

Towards a comprehensive taxonomy of linear structures (over fields)

1. Dual of a vectorspace

equiped with $+ := +_{\mathbb{F}}$, $\cdot := \cdot_{\mathbb{F}}$

=> **Definition:** let $(V,+,\cdot)$ be a $\mathbb{F}$-vector space

Then define a) the set $V^* := \{\varphi : V \to \mathbb{F} \mid \varphi \text{ homo}\}$

$$\left.\begin{array}{l} \text{b) } \oplus : V^* \times V^* \to V^* \\ \odot : \mathbb{F} \times V^* \to V^* \end{array}\right\} \begin{array}{l} \text{pointwise defined} \\ (\varphi \oplus \psi)(v) := \varphi(v) +_{\mathbb{F}} \psi(v) \\ (\lambda \odot \varphi)(v) := \lambda \cdot_{\mathbb{F}} \varphi(v) \end{array}$$

=> **Theorem:** $(V^*, \oplus, \odot)$ is an $\mathbb{F}$-vector space,
called the dual vectorspace $(V,+,\cdot)$

↳ proof:

C
A
N
I
A
D
D
U

=> Examples:

1) $C^\infty(\mathbb{R}) : \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ arbitrary often differentiable}\}$

↳ recall: $' : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ homo
$f \mapsto f'$

now consider $'(0) : C^\infty(\mathbb{R}) \to \mathbb{R}$
$f \mapsto f'(0)$

Question: $'(0) \overset{?}{\in} C^\infty(\mathbb{R})^*$

pointwise def $+ C^\infty(\mathbb{R})$

a) additive: $'(0)(f+g) \overset{\text{def}}{=} (f+g)'(0) \overset{\text{ana}}{=} (f'+g')(0) \overset{\swarrow}{=} f'(0) + g'(0)$

b) scaling: $'(0)(\lambda \cdot f) \overset{\text{def}}{=} \lambda \cdot f'(0)$

2) $\int_0^1 : P_{\mathbb{R}}^n \to \mathbb{R}$

↳ Claim: $\int_0^1 \in P_{\mathbb{R}}^{n*}$!

Upshot $(V, +, \cdot)$ $\mathbb{R}$-V.S. induces $(V^*, \oplus, \odot)$ $\mathbb{R}$-V.S.

=> Remark: velocity at a point is a vector then momentum at that point is an element of the dual space is a "dual vector" / "co-vector"

=> Terminology / jargon: An element of V^* is loosely referred to as "a dual vector" or "a co-vector". Same remarks as for the terminology "vector" apply.

2. Dual of a homomorphism

$$\begin{tikzcd} V \arrow[r, "\xi"] \arrow[dr, "\alpha \circ \xi"'] & W \arrow[d, "\alpha"] \\ & F \end{tikzcd}$$

=> **Definition:** let $V \xrightarrow{\xi} W$ be a homo (dual map) "given"

Then define $V^* \xleftarrow{\xi^*} W^*$ rewritten as $\xi^* : W^* \to V^*$

$$\underset{\alpha}{\overset{\cup\!\!\!\!\!\!\;\;}{}} \mapsto \alpha \circ \xi$$

$\alpha : W \to F$ homo

$V \to F$ homo

↳ Recall: $A_a \xrightarrow{f} B \xrightarrow{g} C$ with $g \circ f$ from A to C ($\circ$:= after)

$(g \circ f)(a) = g(f(a))$ "composition"

=> Example: Consider the homo

$\ln : \mathbb{R}_+ \to \mathbb{R}$

$r \mapsto \ln(r)$

is a linear map

Calculate $\ln^* : \mathbb{R}^* \to \mathbb{R}_+^*$

$$a \mapsto (a \circ \ln)(r) = \alpha(\ln(r))$$
$$= \alpha_a(\ln(r))$$
$$= a \ln(r)$$

=> $\alpha \circ \ln = a \cdot \ln \ldots$

Recall: $(\mathbb{R}, +, \cdot)$ $\mathbb{R}$-VS for $(\mathbb{R}_+, +, \cdot)$

$\alpha_a : \mathbb{R} \to \mathbb{R}$ homo

$x \to a \cdot_{\mathbb{R}} x$ for any $a \in \mathbb{R}$

"Co-vectors eat vectors and spit out fields"

3. Multi-linear maps

=> **Definition:** let $\left.\begin{matrix}(A_1, +_1, \cdot_1) \\ \vdots \\ (A_n, +_n, \cdot_n) \\ \text{and} \\ (V, +, \cdot)\end{matrix}\right\}$ be F-vector spaces

Then any map $\Phi: A_1 \times \ldots \times A_n \to V$

$$(a_1, \ldots, a_n) \mapsto \Phi(a_1, \ldots a_n)$$

is called multi-linear if:

a) Φ is additive, seperately in each cartesian factor :

$$\Phi(a_1 + \tilde{a}_1, a_2, \ldots, a_n) = \Phi(a_1, a_2, \ldots, a_n) + \Phi(\tilde{a}_1, a_2, \ldots, a_n)$$

$$\vdots$$

$$\Phi(a_1, \ldots, a_{n-1}, a_n + \tilde{a}_n) = \Phi(a_1, \ldots, a_{n-1}, a_n) + \Phi(a_1, \ldots, a_{n-1}, \tilde{a}_n)$$

=> key examples:

a) linear maps are multilinear maps

b) Bilinear maps

$B: A \times B \to V$ multilinear maps

c) pseudo-inner products

$g: V \times V \to$, with the symmetry requirement:

$g(v, w) \overset{!}{=} g(w, v)$

d) Tensors over a vectorspace

=> **Definition:** let $(V, +, \cdot)$ be an F-vector space.

A (p,q)-tensor over V is a multilinear map of the special form:

$$T: \underbrace{V^* \times \ldots \times V^*}_{p\text{-factors}} \times \underbrace{V \times \ldots \times V}_{q\text{-factors}} \to F$$

↳ Examples:

a) g is a $(0,2)$-tensor over V

b) $\alpha \in V^*$ ~~claim A: α is a $(1,0)$-tensor over V~~

~~$\alpha: V^* \to F$~~

claim B: α is a $(0,1)$-tensor over V

$\alpha: V \to F$ multi-~~linear~~

"a co-vector is a $(0,1)$-tensor"

$\varphi: V \to V$

$\updownarrow$

$\Phi_\varphi: V^* \times V \to F$

Bases and dimension

1. Definitions

=> **Definition**: let $G \subseteq V$, where $(V, \oplus, \odot)$ is an F-vector space
G is called a generating set of the vectorspace if
$\text{span}(G) = V$.

=> **Definition**: G is called a linearly independent set if:
for any finite subset $\{l_1, \dots, l_n\} \subseteq G$
the homomorphism $\sigma : F^n \to V$

$$(\lambda_1, \dots, \lambda_n) \mapsto \bigoplus_{i=1}^{n} \lambda_i \odot l_i$$

has kernel $\ker \sigma = \{(0, \dots, 0)\}$

=> **Definition**: G basis for the vectorspace if G is both a generating set and a linearly independent set

=> **Remarks:**

1) Every vectorspace has a generating set
↳ proof: Take $G = V$

2) If $0_V \in G$ then G is not linearly independent
↳ proof: take $G = \{0_V\}$

$$\sigma : F \to V$$
$$\lambda \to \lambda \odot 0_V = 0_V$$
$$\Rightarrow \ker \sigma = F \neq \{0_F\}$$ (fields need to have 2 elements)

3) If G is a finite set ($\{G_1, \dots, G_n\}$ for some $m \in \mathbb{N}$)
Then the definition of G being linearly independent collapses to that condition that one single homomorphism

$$\sigma : F^m \to V$$
$$(\lambda_1, \dots, \lambda_m) \mapsto \overbrace{\bigoplus_{i=1}^{m} \lambda_i \odot g_i}^{\text{A linear combination of } g_1, \dots, g_m}$$

has trivial kernel: $\ker \sigma = \{0_{F^m}\}$

"A set can be to small to be a generating set and can become to big to be linearly independent"

=> Example: Claim $\{(0,0,1),(0,1,0),(1,0,0)\}$
is a generation set for F^3
True since any $\underset{\overset{\cap}{F^3}}{(\lambda_1,\lambda_2,\lambda_3)} = \lambda_1 \odot (0,0,1) + \lambda \odot (0,1,0) + \lambda \odot (1,0,0)$

2. Finitely general vectorspaces

=> **Definition:** $(V, \oplus, \odot)$ is called finitely generated if there exists a finite set G that is a generating set for the vectorspace

=> Theorem: Every finitely generated vector space has a basis
↳ proof: 8 8 CANVAS

3. Dual basis for the dual space

consider $(V, \oplus, \odot)$
Let $\{e_1, \dots, e_n\}$ be a basis for V
Let $\{\varepsilon^1, \dots, \varepsilon^n\}$ be a basis for V^* $(V^*, \boxplus, \boxdot) = \{\varphi : V \to F \mid \varphi \text{ homo}\}$

=> Convention: basis elements of V are labelled by downstairs index
basis elements of V^* are labelled by upstairs index

=> Definition: $\{\varepsilon^1, \dots, \varepsilon^n\}$ is called the dual basis with respect to $\{e_1, \dots, e_n\}$

$$\varepsilon^a(e_b) \overset{!}{=} \begin{Bmatrix} 1_F & \text{if } a = b \\ 0_F & \text{else} \end{Bmatrix} =: \delta^a_b \qquad a, b = 1, \dots, n$$

4. Components of tensors w.r.t. a choice of basis

=> recall: a (p,q)-tensor T is a (p,q) multi-linear map

$$T : \underbrace{V^* \times \dots \times V^*}_{p} \times \underbrace{V \times \dots \times V}_{q} \to F$$

Then the components of T with respect to a chosen basis $e_1, \dots, e_n \overset{\text{unique}}{\rightsquigarrow} \varepsilon^1, \dots, \varepsilon^n$ are the field valued numbers

$$T^{a_1, \dots, a_p}{}_{b_1, \dots, b_q} := T(\varepsilon^{a_1}, \dots, \varepsilon^{a_p}, e_{b_1}, \dots, e_{b_q}) \qquad a_1, \dots, a_p, b_1, \dots, b_q = 1, \dots d$$

=> Example:

Let's consider $(1,1)$-tensor $\varphi : V^* \times V \to F$ multi-linear map

$V = \text{span}\{e_1, \dots, e_d\}$ ← choose this

$V^* = \text{span}\{\varepsilon^1, \dots, \varepsilon^d\}$

$\underbrace{\varphi^a{}_b}_{d^2 \text{ many numbers}} := \varphi\{\varepsilon^a, e_d\} \qquad a, b = 1, \dots, d$

linear dependence

Observe: $v \in V \Rightarrow \exists!\, v^1, \dots, v^d \in F : V := \bigoplus_{i=1}^{d} \underset{\overset{\cap}{F}}{v^i} \odot \underset{\overset{\cap}{V}}{e_i}$

Proof of uniqueness:

Suppose $v = \bigoplus_{i=1}^{d} v^i \odot e_i$, $v = \bigoplus_{i=1}^{d} \tilde{v}^i \odot e_i$

$$\Rightarrow 0_V = \bigoplus_{i=1}^{d} (v^i - \tilde{v}^i) \odot e_i \Rightarrow v^i = \tilde{v}^i$$

Similarly for

$\sigma \in V^* \Rightarrow \exists \sigma^1, \dots, \sigma^d \in F : \sigma = \bigoplus_{i=1}^{d} \sigma_i \odot \varepsilon^i$

Then: $f(\sigma, v) = f(\bigoplus_{i=1}^{d} \sigma_i \odot \varepsilon^i, \bigoplus_{j=1}^{d} v^j \odot e_j)$

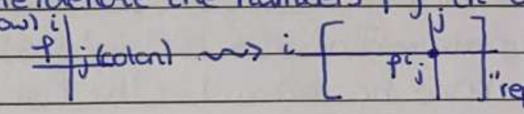
add of f in 1st slot $= \sum_{i=1}^{d} f(\sigma_i \odot \varepsilon^i, \bigoplus_{j=1}^{d} v^j \odot e_j)$

$= \sum_{i=1}^{d} \sum_{j=1}^{d} f(\sigma_i \odot \varepsilon^i, v^j \odot e_j) = \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_i \, f(\varepsilon^i, v^j \odot e_j)$

$= \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_i \cdot v^j \cdot f(\varepsilon^i, e_j) = \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_i \cdot v^j \cdot f^i{}_j$

⇒ Some wild conventions:

One may choose to store/denote the numbers $f^i{}_j$ in a Square arrangement



"representing matrix"

$V^* \times V \to \mathbb{R}$	$f^i{}_j$
$V \times V \to \mathbb{R}$	f_{ij}
$V^* \times V^* \to \mathbb{R}$	f^{ij}
$V \times V^* \to \mathbb{R}$	$f_i{}^j$

⇒ lemma: let $\{s_1, \dots, s_n\} \subseteq V$ be a linearly independent, non-empty finite subset of an F-vector space $(V, \oplus, \odot)$. Then for any $v \in V$ with $v \notin \{s_1, \dots, s_n\}$, the set $\{s_1, \dots, s_n, v\}$ is linearly dependent if and only if $v \in \operatorname{span}(\{s_1, \dots, s_n\})$

↳ proof. Assume the set $\{s_1, \dots, s_n, v\}$ is linearly dependent, then it contains elements $u_0, \dots, u_\ell$ for some $\ell \in \mathbb{N}$ with $0 \leq \ell \leq n$ such that $\bigoplus_{i=0}^{\ell} \lambda_i \odot u_i = 0_V$ with all $\lambda_0, \dots, \lambda_i \in F^*$.

Since $\{s_1, \dots, s_n\}$ is linear independent, one element of the set $\{u_0, \dots, u_\ell\}$, say u_0 must be v, while all others are elements of $\{s_1, \dots, s_n\}$. But then $v = -\bigoplus_{i=1}^{\ell} (\lambda_i / \lambda_0) \odot u_i$, whence $v \in \operatorname{span}(\{s_1, \dots, s_n\})$

Converse

=> Theorem: Every finitely generated vector space has a bases

↳ proof: Let $S \subseteq V$ be a finite set that generates V.
We show that then some subset of S is a basis.

If $S = \emptyset$ or $S = \{0_V\}$, then $\emptyset$ is a basis and $V = \{0_V\}$.

Otherwise there is an element $s_1 = 0_V$ in S, so that the set $\{s_1\}$ is linearly independent.
Continue choosing, as long as possible, further elements $s_2, \dots, s_k \in S$ such that $\{s_1, \dots, s_k\}$ is linearly independent. This process can only end at some natrual $k \leq |S|$ in either one of two ways.

1) One has exhausted the finite set S and thus has shown that S is a linearly independent set. Since S is a generating set it is then also a basis of V.
2) one finds that every remaining element $s \in S \setminus \{s_1, \dots, s_k\}$ renders $\{s_1, \dots, s_k, s\}$ linearly dependent. But then any such $s \in \text{span}(\{s_1, \dots, s_k\})$, according to the preceding lemma. Thus $S \subseteq \text{span}(\{s_1, \dots, s_k\})$. But then since $S \subseteq T$ implies $\text{span}(S) \leq \text{span}(T)$, on the one hand, and $\text{span}(U) = U$ for any subspace $U \leq V$ in general and thus for $U = \text{span}(\{s_1, \dots, s_k\})$ in particular, on the other hand we have $V \leq \text{span}(\{s_1, \dots, s_k\}) \leq V$, whence the linearly independent set $\{s_1, \dots, s_k\}$ is also a generating set and thus a basis of V.

Steinitz exchange lemma

=>Theorem: Let $\{g_1,\dots,g_G\}$ be a generating set and $\{l_1,\dots,l_L\}$ be a linearly independent set for a finitely generated $\mathbb{F}$-vector space$(V,\oplus,\odot)$ Then $L \leq G$ and the elements of $\{g_1,\dots,g_G\}$ may be relabeled such that $\{l_1,\dots,l_L,g_{L+1},\dots,g_G\}$ is a generating set.

↳proof: By induction on L
For the induction start $L=0$, the result holds trivially.

Suppose the result has been shown for $L-1$, so that $\{l_1,\dots,l_{L-1},g_L,\dots,g_G\}$ is a generating set after relabeling all elements of G.
Then l_L can be written as:

$$l_L = \bigoplus_{i=1}^{L-1} \lambda_i \odot l_i \oplus \bigoplus_{j=1}^{G} \lambda_j \odot g_j$$

where some of the $\lambda_L,\dots,\lambda_G$ must be non-zero (and thus $L \leq G$). For otherwise the occurence of $1 \odot l_L$ on the left hand side would contradict the assumed linear independence of the set $\{l_1,\dots,l_L\}$.
After an appropiate relabelling of $g_L,\dots,g_G$, we may then assume, in particular, that $\lambda_L \neq 0_{\mathbb{F}}$
But then:

$$g_L = \lambda_L^{-1}(l_L \oplus (-\bigoplus_{i=1}^{L-1} \lambda_i \odot l_i) \oplus (-\bigoplus_{j=L+1}^{G} \lambda_j \odot g_j))$$

So that $\{l_1,\dots,l_{L-1},g_L,\dots,g_G\} \subset \text{span}(\{l_1,\dots,l_L,g_{L+1},\dots,g_G\}) = V$

Two + epsilon good ways to deal with linear structures

=> Let $(V,\oplus,\odot)$ be an F-vector space ⇝ $(V^*,\boxplus,\boxdot)$ finite dimension

$e_1,\dots,e_d$ "basis"

$\varepsilon^1,\dots,\varepsilon^d$ "Dual basis" : $\varepsilon^a(e_a)=\delta^a_b$

↳ kronecker delta: $\delta^a_b := \begin{cases} 1_F \text{ if } a=b \\ 0_F \text{ if } a\neq b \end{cases}$

$$\varepsilon^a(v)=\varepsilon^a\left(\bigoplus_{i=1}^{d} v^i\odot e_i\right) = \sum_{i=1}^{d} v^i\cdot \underbrace{\varepsilon^a_i}_{\delta^a_i} = v^a\cdot 1_F = v^a$$

=> Theorem: Let $v\in V$, $\sigma\in V^*$, then:

i) $v=\bigoplus_{m=1}^{d} \varepsilon^m(v)\odot e_m$

↳ proof: Since $e_1,\dots,e_d$ is a basis for V, we know that there are unique numbers $v^1,\dots,v^d\in F$ such that

$v=\bigoplus_{m=1}^{d} v^m\odot e_m$

Now calculate $\varepsilon^m(v) = \text{(above)} = v^m$ ∘∘∘∘ Components with respect to the basis $e_1,\dots,e_d$

ii) $\sigma=\boxplus_{m=1}^{d} \sigma(e_m)\boxdot\varepsilon^m$

↳ Since $e_1,\dots,e_d$ is a basis for V, we know that there exists a unique dual basis $\sigma_1,\dots,\sigma_d\in F$ such that

$\sigma=\boxplus_{m=1}^{d} \sigma_m\boxdot\varepsilon^m$

Now calculate $\sigma(e_m) = \left(\boxplus_{m=1}^{d} \sigma_m\boxdot\varepsilon^m\right)e_m$

$= \sum_{m=1}^{d} \sigma_m\cdot\varepsilon^m{}_m$

$= \sigma_m\cdot 1_F = \sigma_m$

=> There are three ways to think about/do calculations in the field of linear structures

↳ See next pages

Object	Components w.r.t a choosen basis	Cemeteries uplabeling → row, downlabeling → column
$v \in V$	$\Leftrightarrow v^a := \varepsilon^a(v) \in F$ $\quad a=1,\dots,d$	$\Leftrightarrow \begin{bmatrix} v^1 \\ \vdots \\ v^d \end{bmatrix}$ a column matrix/vector
$\sigma \in V^*$	$\Leftrightarrow \sigma_a := \sigma(e_a) \in F$ $\quad a=1,\dots,d$	$\Leftrightarrow [\sigma_1, \dots, \sigma_d]$ a row matrix/vector
$\varphi : V \to V$	$\Leftrightarrow \varphi^a{}_b := \varepsilon^a(\varphi(e_b)) \in F$ $\quad a=1,\dots,d$ $b=1,\dots,d$	$\Leftrightarrow \begin{array}{cc} \text{,column } b & \\ \begin{bmatrix} \varphi^1{}_1 & & \varphi^1{}_2 & \cdots & \varphi^1{}_d \\ \vdots & \varphi^a{}_b & & & \vdots \\ \varphi^d{}_1 & & \varphi^d{}_2 & \cdots & \varphi^d{}_d \end{bmatrix} & \text{row } a \end{array}$
$\varphi(v) \in V$	$(\varphi(v))^a \overset{\text{line 1}}{=} \varepsilon^a(\varphi(v))$ $= \varepsilon^a(\varphi(\bigoplus_{m=1}^{d} v^m \odot e_m))$ $\Leftrightarrow \overset{\text{add of } \varphi, \varepsilon^a}{=} \sum_{m=1}^{d} \varepsilon^a(\varphi(v^m \odot e_m))$ $\overset{\text{scaling of } \varphi, \varepsilon^a}{=} \sum_{m=1}^{d} v^m \cdot \overbrace{\varepsilon^a(\varphi(e_m))}^{\varphi^a{}_m}$ $= \sum_{m=1}^{d} v^m \cdot_F \varphi^a{}_m \overset{c}{=} \sum_{m=1}^{d} \varphi^a{}_m \cdot v^m$	$\Leftrightarrow \begin{bmatrix} (\varphi(v))^1 \\ \vdots \\ (\varphi(v))^d \end{bmatrix} = \begin{bmatrix} \varphi^1{}_1 & \cdots & \varphi^1{}_d \\ \vdots & & \vdots \\ \varphi^d{}_1 & \cdots & \varphi^d{}_d \end{bmatrix}$ ❀ $\begin{bmatrix} v^1 \\ \vdots \\ v^d \end{bmatrix}$ ❀ : row times column (correspons to endo applied to vector
$\varphi^* : V^* \to V^*$ $\varphi^*(\tau) := \tau \circ \varphi \Leftrightarrow$ $\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[dr, "\varphi^*"'] & V \arrow[d, "\tau"] \\ & F \end{tikzcd}$	$\varphi^{*a}{}_b := (\varphi^*(\varepsilon^a))(e_b)$ $= (\varepsilon^a \circ \varphi)(e_b)$ $\overset{\text{def} \circ}{=} \varepsilon^a(\varphi(e_b))$ $= \varphi^a{}_b$	
$\varphi^*(\tau) \in V$	$(\varphi^*(\tau))_a := (\varphi^*(\tau))(e_b) = (\tau \circ \varphi)(e_b)$ $= \tau(\varphi(e_b)) = \tau(\varphi(e_b))\varepsilon^m$ $\Leftrightarrow \quad = \tau^m \varphi^m{}_b$	$[(\varphi^*(\tau))_1, \dots, (\varphi^*(\tau))_d] = [\tau_1, \dots, \tau_d]$ ❀ $\begin{bmatrix} \varphi^1{}_1 & \cdots & \varphi^1{}_d \\ \vdots & & \vdots \\ \varphi^d{}_1 & \cdots & \varphi^d{}_d \end{bmatrix}$ $\Leftrightarrow$

$\sigma(v) \in F$

$$\sigma(v) = \left(\boxplus_{m=1}^{d} \sigma_m \boxdot \varepsilon^m\right)\left(\bigoplus_{n=1}^{d} v^n \odot e_n\right)$$

$$= \sum_{m=1}^{d} (\sigma_m \cdot \varepsilon^m) \sum_{n=1}^{d} (v^n \cdot e_n)$$

$$\overset{\dot{c}}{=} \sum_{m=1}^{d} \sum_{n=1}^{d} \sigma_m \cdot \varepsilon^m \cdot v^n \cdot e_n$$

$$= \sum_{m=1}^{d} \sigma_m(e_n) \cdot \varepsilon^m(v^n)$$

$\Leftrightarrow$

$$= \sum_{m=1}^{d} \left(\boxplus_{m=1}^{d} \sigma_m \boxdot \varepsilon^m\right) e_n \left(\bigoplus_{m=1}^{d} v^n \odot e_n\right) \varepsilon^m$$

$$= \sum_{m=1}^{d} \left(\sum_{m=1}^{d} \sigma_m \cdot \varepsilon^m{}_n\right)\left(\sum_{m=1}^{d} v^m \cdot \varepsilon^m{}_n\right)$$

$$= \sum_{m=1}^{d} (\sigma_m \cdot 1_F) \cdot (v^m \cdot 1_F)$$

$$= \sum_{m=1}^{d} \sigma_m \cdot v^m$$

$$\sigma(v) = [\sigma_1, \ldots, \sigma_d] \circledast \begin{bmatrix} v^1 \\ \vdots \\ v^d \end{bmatrix}$$

$\Leftrightarrow$

$$\begin{tikzcd} V \arrow[r, "\beta"] \arrow[dr, "\alpha\circ\beta"'] & V \arrow[d, "\alpha"] \\ & V \end{tikzcd}$$

$$(\alpha\circ\beta)^a{}_b = \varepsilon^a((\alpha\circ\beta)(e_b))$$

$$= \varepsilon^a(\alpha(\underbrace{\beta(e_b)}_{\substack{\| \\ \bigoplus_{m=1}^{d} \varepsilon^m(\beta(e_b)\odot e_m)}}))$$

$\Leftrightarrow$

$$= \sum_{m=1}^{d} \beta^m{}_b \cdot \varepsilon^a(\alpha(e_m))$$

$$= \sum_{m=1}^{d} \alpha^a \cdot \beta^m{}_b$$

$$\Leftrightarrow \begin{bmatrix} \alpha\circ\beta^1{}_1 & \cdots & \alpha\circ\beta^1{}_d \\ \vdots & & \vdots \\ \alpha\circ\beta^d{}_1 & \cdots & \alpha\circ\beta^d{}_d \end{bmatrix} = \begin{bmatrix} \alpha^1{}_1 & \alpha^1{}_d \\ \alpha^d{}_1 & \alpha^d{}_d \end{bmatrix} \circledast \begin{bmatrix} \beta^1{}_1 & \cdots & \beta^1{}_d \\ \vdots & & \vdots \\ \beta^d{}_d & \cdots & \beta^d{}_d \end{bmatrix}$$

$g: V \times V \to F$

$g(v,w) \in F$

$$g_{ab} := g(e_a, e_b) \in F$$

$$g(v,w) = \ldots = \sum_{m,n=1}^{d} \cdot v^m \cdot g_{m,n} \cdot w^n$$

$\Leftrightarrow$

$$= \sum_{m=1}^{d} \left(\sum_{n=1}^{d} g_{m,n} \cdot w^n\right) v^m$$

$$[[g_{11}, \ldots, g_{1d}], \ldots, [g_{d1}, \ldots, g_{dd}]]$$

$$\left([[g_{11}, \ldots, g_{1d}], \ldots, [g_{d1}, \ldots, g_{dd}]]\begin{bmatrix} w^1 \\ \vdots \\ w^d \end{bmatrix}\right)\begin{bmatrix} v^1 \\ \vdots \\ v^d \end{bmatrix}$$

$$\Leftrightarrow \overset{?}{=} [(g_{11}w^1 + \ldots + g_{1d}w^d), \ldots, (g_{d1}w^1 + \ldots + g_{dd}w^d)]\begin{bmatrix} v^1 \\ \vdots \\ v^d \end{bmatrix}$$

$$= g_{11}w^1v^1 + g_{12}w^2v^2 + \ldots + g_{dd}w^dv^d$$

Tekstbooks do this

$g: V \times V \to F$ $g_{a,b} := g(e_a, e_b)$

$\Leftrightarrow$

$g(v,w)$ $g(v,w) \cdot \sum_m \sum_n = g_{mn} v^m w^n$

$$\begin{bmatrix} g_{1,1}, g_{1,2} & \cdots & g_{1d} \\ \vdots & & \vdots \\ g_{d1} & \cdots & g_{dd} \end{bmatrix}$$

$$\begin{bmatrix} v^1 \\ \vdots \\ v^d \end{bmatrix}^T \circledast \begin{bmatrix} g_{11} & \cdots & g_{1d} \\ \vdots & & \vdots \\ g_{d1} & \cdots & g_{dd} \end{bmatrix} \circledast \begin{bmatrix} w^1 \\ \vdots \\ w^d \end{bmatrix}$$ ill-defined

A message from our sponsor

=> It is straight foward to calculate the image $f(v)$ of some $v \in V$ under a given homomorphism $f: V \to W$

1. The inverse problem

=> The inverse problem is to find all those $v \in V$, which for a given $b \in W$ and a given homomorphism $f: V \to W$ yield $\underset{\nwarrow}{f}(v) = \underset{\uparrow}{b}$
given

=> Remark: It will turn out that there are three different classes of solution spaces for the inverse problem

=> Definition: The solution space $S_{f(v)=b} := \{v \in V \mid f(v) = b\}$

class 1) $S = \emptyset$ (no solution)
class 2) $S = \{v\}$ (unique solution)
class 3) S infinite set

2. Conversion into a mere numbers game

on F

=> For finite dimensional vector spaces V and W,
choose basis $e_1, \dots, e_n$ for V (n = dimension V)
choose basis $d_1, \dots, d_m$ for W (m = dimension W)
unique dual basis $\varepsilon^1, \dots, \varepsilon^n$ for V^*
and $\delta^1, \dots \delta^m$ for W^*

Thus $f(v) = b$

$a, b = 1, \dots, n$
$A, B = 1, \dots, m$

$\in W^*$ $\quad$ $\in W$ $\quad$ $v = \bigoplus_{b=1}^{n} \underbrace{\varepsilon^b(v)}_{= v^b} e_b$ $\quad$ b^A

$$\delta^A(f(v)) = \delta^A(b)$$

$$\|$$

$$\delta^A(f(\bigoplus_{b=1}^{n} v^b e_b)) = \delta^A(\boxplus_{b=1}^{n} v^b \boxdot f(e_b)) = \sum_{b=1}^{n} v^b \overline{\delta^A(f(e_b))}$$

$f^A{}_b \in F$

Thus $f(v) = b \Leftrightarrow \forall A = 1, \dots, m : \sum_{b=1}^{n} f^A{}_b \cdot v^b = b^A$

add. on F $\quad$ $\in F$ $\quad$ mult. on F $\quad$ $\in F$

=> Convention: (Einstein)

In Linear structures (and anywhere where they emerge in advanced context) any equation that is written in components. an index a that appears once up and once down inevitably comes with a sum over it (if indeed one starts from an abstract expression). Einstein says: write $\overset{\cdots}{\sum}$ in invisible ink.

↳ 2 indices down is not possible

↳ $\sum_{b=1}^{n} f^{1}{}_{b} v^{b} = b^{1} \Leftrightarrow f^{1}{}_{b} v^{b} = b^{1}$

$$\Leftrightarrow \begin{bmatrix} f^1{}_1 & \cdots & f^1{}_m \\ \vdots & & \vdots \\ f^n{}_1 & \cdots & f^n{}_m \end{bmatrix} \circledast \begin{bmatrix} v^1 \\ \vdots \\ \vdots \\ v^m \end{bmatrix} = \begin{bmatrix} b^1 \\ \vdots \\ b^n \end{bmatrix}$$

here $n<m$

$\circledast$: plimes (plus, times)

3. The row echelon form

=> Remark: this is a form for the mxn matrix/cemetry above from which one readily read off the solution spaces S.

=> Definition: The pivot element of some given row is the left-most non-zero ~~element~~ component in the row. If there is no such, there is no pivot.

=> Definition: A mxn matrix/cemetry is said to be in row echelon form (REF) if:

1) All rows without a pivot are below all rows with a pivot.
2) The pivot of any row (that has a pivot) is to the right of every pivot of a preceding row.

=> Examples: $\begin{bmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&0&2&0 \end{bmatrix}$ ✗ $\begin{bmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&2&0&0 \end{bmatrix}$ ✗ $\begin{bmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \end{bmatrix}$ ✓ stairs

4. Three elementary "moves" to achieve REF

=> Strategy $\underset{\overset{\uparrow}{\text{given}}}{[\,\vdots\,]} \begin{bmatrix}\vdots\\ \vdots\end{bmatrix} \overset{\leftarrow \text{ desired}}{=} [\,\vdots\,] \leftarrow$ given

$f^A{}_b v^b = b^A$
$\Updownarrow$
$\mu^B{}_A f^A{}_b v^b = \mu^B{}_A b^A$

=> Definition: $(\mu(I,\lambda))^B{}_A$ "Scales row I by λ"

$$\begin{array}{c|ccccc|}
 & & & I & & \\
 & 1 & & & & \\
 & & \ddots & & & \\
I & & & \lambda & & \\
 & & & & \ddots & \\
 & & & & & 1
\end{array}{}^{B}{}_{A}$$

(all empty slots are 0)
$\lambda \in \mathbb{F}^*$

↳ Example: $\underbrace{\begin{bmatrix}1 & 0\\ 0 & \pi\end{bmatrix}}_{\mu(2,\pi)} \underbrace{\begin{bmatrix}2 & 3\\ 4 & 6\end{bmatrix}}_{f} \underbrace{\begin{bmatrix}v_1\\ v_2\end{bmatrix}}_{v} = \begin{bmatrix}1 & 0\\ 0 & \pi\end{bmatrix} \underbrace{\begin{bmatrix}3\\ 8\end{bmatrix}}_{b}$

$$\begin{bmatrix}2 & 3\\ 4\pi & 6\pi\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}3\\ 8\pi\end{bmatrix}$$

↳ Observe: $(\mu(I,\lambda^{-1}))^C{}_B\,(\mu(I,\lambda))^B{}_A = \delta^C{}_A$

=> Definition: $(\alpha(I,\lambda,J))^B{}_A$ "adds $\lambda \odot$ row J to row I"

$$\begin{array}{c|ccc|}
 & 1 & & \\
 & & \ddots & \\
I & \lambda & & 1 \\
 & J & &
\end{array}{}^{B}{}_{A}$$

↳ eg $\begin{bmatrix}1 & 0\\ \lambda & 1\end{bmatrix}\begin{bmatrix}2 & 3 & 10\\ 4 & 6 & -7\end{bmatrix} = \begin{bmatrix}2 & 3 & 10\\ 4+2\lambda & 6+3\lambda & -7+10\lambda\end{bmatrix}$

↳ observation: $(\alpha(I,-\lambda,J))^B{}_C\,(\alpha(I,\lambda,J))^C{}_A = \delta^B{}_A$

=> Definition: $(\chi(I,J))^B{}_A$ "exchanges row I ↔ row J"

$$\begin{array}{c|cccccc|}
 & 1 & & & & & \\
 & & 1 & & & & \\
I & & & 0 & 1 & & \\
J & & & 1 & 0 & & \\
 & & & & & 1 & \\
 & & & & & & 1 \\
 & & & I & J & &
\end{array}{}^{B}{}_{A}$$

↳ example: $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}2 & 3 & 10\\ 4 & 6 & -7\end{bmatrix} = \begin{bmatrix}4 & 6 & -7\\ 2 & 3 & 10\end{bmatrix}$

↳ observation: $(\chi(I,J))^B{}_C\,(\chi(I,J))^C{}_A = \delta^B{}_A$

=> upshot: α, μ, χ effects equivalence transformations of the original $\varphi^{1}b\, v^{b} = b^{1}$

↳ strategy: to take $\varphi^{1}b$ to REF: Apply α, μ, χ in an appropiate order to achieve the goal.

=> Examples:

a) Linear system without solution

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$\alpha(2,-2,1) \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}}_{\text{REF}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Leftrightarrow \begin{array}{l} 2x + 3y = 3 \\ 0 = 2 \end{array}$$

$$\Rightarrow S = \emptyset$$

b) Linear system with a unique solution

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$\Leftrightarrow \underbrace{\begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}}_{\text{REF}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Leftrightarrow \begin{array}{l} 2x + 3y = 3 \\ -y = 2 \end{array}$$

$$\Rightarrow y = -2, \ x = \frac{9}{2}$$

$$\Rightarrow S = \left\{ \underbrace{\begin{bmatrix} 9/2 \\ -2 \end{bmatrix}}_{v^{b}} \right\} \qquad v = v^{b} e_{b} = \frac{9}{2} \odot e_1 \oplus (-2) \odot e_2$$

$$S = \left\{ \frac{9}{2} \odot e_1 \oplus (-2) \odot e_2 \right\}$$

c) Linear system with ∞ many solutions

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \Leftrightarrow \begin{array}{l} 2x + 3y = 3 \\ 0 = 0 \end{array} \Leftrightarrow \overset{\substack{\text{Pivot column} \\ \downarrow}}{x} = \frac{3}{2}(1 - \overset{\substack{\text{non-pivot column} \\ \downarrow}}{y})$$

$$\text{Somehow see: } S = \left\{ \begin{bmatrix} \frac{3}{2}(1-s) \\ s \end{bmatrix} \,\middle|\, s \in \mathbb{R} \right\}$$

$$S = \left\{ \frac{3}{2}(1-s) \odot e_1 \oplus s \odot e_2 \right.$$

$$\begin{array}{cc} \text{Pivot column} & \text{non-pivot column} \\ \downarrow & \downarrow \\ 2 & 3 \\ 0 & 0 \end{array}$$

e.g. $e_1 = (x \mapsto 1 + x^2)$

$e_2 = (x \mapsto 7)$

$$S = \left\{ (x \mapsto \frac{3}{2}(1-s) \odot 1 + x^2) \oplus (x \mapsto 7s) \right\}$$

$$= \left\{ x \mapsto \frac{3}{2}(1-s) \cdot (1 + x^2) + 7s \right\}$$

=> **Definition:** A column of a matrix/cemetry in REF is called a pivot column if it contains a pivot element of the REF, otherwise it is called a non-pivot column

Augmented matrices/cemetries

1. Last lesson

=> The inverse problem

Let $e_1,\dots,e_n$ be a basis for V, get dual-basis $\varepsilon^1,\dots,\varepsilon^n$

Let $g_1,\dots,g_n$ be a basis for W, get dual-basis $\gamma^1,\dots,\gamma^m$

Given $\varphi: V \to W$ homomorphism

$$\varphi^i{}_j = \gamma^i(\varphi(e_j)) \text{ also } b \in W \quad b^i = \gamma^i(b)$$

Problem: find $v \in V$ such that $\varphi(v) = b$

Solution: $v = \sum_{j=1}^{n} \varepsilon^j(v) e_j$

$$\text{then } b = \varphi(v) = \varphi\left(\sum_{j=1}^{n} \varepsilon^j(v) e_j\right) = \sum_{j=1}^{n} \varepsilon^j(v)\, \varphi(e_j)$$

$$\text{therefore: } b^i = \gamma^i(b)\left(\sum_{j=1}^{n} \varepsilon^j(v)\, \varphi(e_j)\right) = \sum_{j=1}^{n} \varepsilon^j(v)\, \gamma^i(\varphi(e_j))$$

$$= \sum_{j=1}^{n} \varepsilon^j(v)\, \varphi^i{}_j$$

We can package this:

$$(S) \quad \begin{bmatrix} \varphi^1{}_1 & \cdots & \varphi^1{}_n \\ \vdots & & \vdots \\ \varphi^m{}_1 & \cdots & \varphi^m{}_n \end{bmatrix} \begin{bmatrix} \varepsilon^1(v) \\ \vdots \\ \vdots \\ \varepsilon^n(v) \end{bmatrix} = \begin{bmatrix} b^1 \\ \vdots \\ b^m \end{bmatrix}$$

=> key words:

REF: row echelon form

elementary cemeteries: $[\mu(i,\lambda)]$, $[\alpha(i,\lambda,j)]$, $[\chi(i,j)]$

2. Augmented matrices/cemetries

=> **Definition:** The augmented matrix/cemetery of S is

$$(S'): \left[\begin{array}{ccc|c} \varphi^1{}_1 & \cdots & \varphi^1{}_n & b^1 \\ \vdots & & \vdots & \vdots \\ \varphi^m{}_1 & \cdots & \varphi^m{}_n & b^m \end{array}\right]$$

=> **Definition:** An elementary row on (S') is an operation of any of the following types:

(a) Adding a scaling of a row to the other (α)

(b) Scaling of a row by a non-zero number (μ)

(c) Interchanging two rows (χ)

=> idea: we can solve linear systems by applying row opperations to S' and bring things to a simpler form (REF, RREF)

↳ Final step: convert things to a system

=> Example: $\left.\begin{matrix} 2x+4y=2 \\ 3x+5y=1 \end{matrix}\right\} \Rightarrow \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ / $\mathbb{R}$

$$\left[\begin{array}{cc|c} 2 & 4 & 2 \\ 3 & 5 & 1 \end{array}\right] \xrightarrow{\mu(1,\frac{1}{2})} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 5 & 1 \end{array}\right] \xrightarrow{\alpha(2,-3,1)} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{\mu(2,-1)} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{\alpha(1,-2,2)} \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \end{array}\right] \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$\Rightarrow x = -3$

$y = 2$

=> Definition: A linear system is called consistent if it has at least one solution, otherwise we call this inconsistent

=> Definition: A matrix/cemetery is said to be in row reduces echelon form (RREF) if:

i) Rows without a pivot are below rows with a pivot

ii) The pivot of each row is to the right of all rows above it.

iii) Each pivot equals 1 and is the only-zero element in it's column

=> Theorem: A linear system (S') is consistent <=> the RREF of the augmented matrix/cemetery has no pivot element in its last column (the RREF of the augmented cemetery has no row of the form $[0\,0 - 0 | b]$ with $b \neq 0$]

3. Relate RREF, dimension ker f, dimension im f

=> Lemma: let V be a finite dimensional vector space and $W \leq V$

Then the $\dim(V/W) = \dim(V) - \dim(W)$

↳ proof: let $w_1, \ldots, w_n$ be a basis for W $(\dim(W) = n)$

and $v_1, \ldots, v_n$ be a basis for V $(\dim(V) = d)$

By steinitz exchange lemma: the elements $v_1, \dots, v_d$ can be relabelled such that $w_1, \dots, w_n, v_{n+1}, \dots, v_d$ is a basis for V.

claim: $[v_{n+1}], \dots, [v_d]$ is a basis for V/W

i) Need to show $[v_{n+1}], \dots, [v_d]$ is linearly independent

Let $\lambda^{n+1}, \dots, \lambda^d \in F$ such that $\bigoplus_{j=n+1}^{d} (\lambda^j \odot [v_j]) = 0_{V/W}$

we know $0_{V/W} = [0_V]$

$\bigoplus_{j=n+1}^{d} (\lambda^j \odot [v_j]) = \bigoplus_{j=n+1}^{d} [\lambda^j \odot v_j] = [\bigoplus_{j=n+1}^{d} (\lambda^j \odot v_j)] = [0_V]$

$\Rightarrow \bigoplus_{j=1}^{d} (\lambda^j \odot v_j) - 0_V \in W$

because $w_1, \dots, w_n$ is a basis for W: $\exists \lambda^1, \dots, \lambda^n \in F$

such that $\bigoplus_{j=n+1}^{d} (\lambda^j \odot w_j) = \bigoplus_{j=n+1}^{n} (\lambda^j \odot w_j)$

$\Rightarrow -\bigoplus_{j=1}^{n} (\lambda^j \odot w_j) \bigoplus_{j=n+1}^{d} (\lambda^j \odot v_j) = 0_V$

$[(-\lambda^1) \odot w_1] \oplus \oplus [(-\lambda^n) \odot w_n] \oplus [\lambda^{n+1} \odot v_{n+1}] \oplus \dots \oplus [\lambda^d \odot v_d] = 0_V$

Since $w_1, \dots, w_n, v_{n+1}, \dots, v_d$ is a basis for V

$-\lambda^1 = -\lambda^2 = \dots = -\lambda^n = \underline{0 = \lambda^{n+1} = \dots = \lambda^d}$

$\Rightarrow [v_{n+1}], \dots [v_d]$ is linearly independent

ii) Need to show $[v_{n+1}], \dots, [v_d]$ is a generating set

Let $u \in V/W$ then $\exists v \in V$ such that $u = [v]$

Since $w_1, \dots, w_n, v_{n+1}, \dots, v_d$ for V, $\exists \lambda^1, \dots, \lambda^d \in F$ such that

$v = \bigoplus_{i=1}^{n} (\lambda^i \odot w_i) \oplus \bigoplus_{j=n+1}^{d} (\lambda^j \odot v_j)$

$[v] = [\bigoplus_{i=1}^{n} (\lambda^i \odot w_i) \oplus \bigoplus_{j=n+1}^{d} (\lambda^j \odot v_j)] = [\bigoplus_{j=n+1}^{d} (\lambda^j \odot v_j)]$

$= \bigoplus_{j=n+1}^{d} (\lambda^j \odot [v_j])$

$\Rightarrow [v_{n+1}], \dots, [v_d]$ is a generating set, thus a basis

$\Rightarrow$ Theorem: let $f : V \rightarrow W$ be a homomorphism and V is finite dimensional. Then dimension $V = \dim(\ker f) + \dim(\operatorname{im} f)$

↳ proof: recall: $V/\ker f \cong \operatorname{im} f$

then $\dim(V/\ker) = \dim \operatorname{im} f$

by lemma: $\dim(V/W) = \dim(V) - \dim(\ker f)$

$\Rightarrow \dim(\operatorname{im} f) = \dim(V) - \dim(\ker f)$

$\dim(V) = \dim(\operatorname{im} f) + \dim(\ker f)$

$\Rightarrow$ Theorem: $\dim(\ker f)$ is equal to the number of zero-row's of an echelon form of $[f^j{}_i]$

↳ If $v \in \ker f$, then $f(v) = 0$

$$(S') \left[\begin{array}{ccc|c} f^1{}_1 & \cdots & f^1{}_m & 0 \\ \vdots & & \vdots & \vdots \\ f^n{}_1 & \cdots & f^n{}_m & 0 \end{array}\right]$$

=> Theorem: $\dim(\operatorname{im} \varphi)$ is equal to the number of non-zero rows of an echelon form of $[\varphi^i{}_j]$

=> **Definition**: $\operatorname{rank}([\varphi^i{}_j]) := \dim \operatorname{im} \varphi$

Determinant of a bilinear form ($V \times V \to F$) is fake news

Determinant of an endomorphism

=> Determinant of an endomorphism on a finite dimension vector space is a canonically defined field element for each endomorphism. Useful in many context.

1. Forms on a vector space

=> **Definition**: A p-form Γ on an F-vector space $(V, \oplus, \odot)$ is a $(0,p)$-tensor $\Gamma : \underbrace{V \times \dots \times V}_{p\text{-times}} \to F$ that is totally anti symmetric.

i.e. for any $v_1, \dots, v_p \in V$ $\Gamma(v_1, \dots, v_i, \dots, v_j, \dots, v_p)$

$= -\Gamma(v_1, \dots, v_j, \dots, v_j, \dots, v_p)$ for any $i \neq j$ in $\underset{1,\dots,p}{\text{range}}$

Define: $\Lambda^p(V) := \{\Gamma \in \dots \mid \Gamma \text{ p-form}\}$

$\oplus, \odot$ pointwise defined

Fact: $(\Lambda^p(V), \oplus, \odot)$ is an F-vector space

"Vectorspace of p-forms over the vector space V"

=> Remarks:

1) A 0-form is an element of F
2) A 1-form is a co-vector
3) For any p-form with $p \geq 2$:

$\Gamma(v_1, \dots, v, \dots, v, \dots, v_p) = -\Gamma(v_1, \dots, v, \dots, v, \dots, v_p)$

$\Leftrightarrow 2\,\Gamma(v_1, \dots, v, \dots, v, \dots, v_p) = 0_F$

$\overset{!!}{\Downarrow}$

$\Gamma(v_1, \dots, v, \dots, v, \dots, v_p) = 0_F$ if $\operatorname{char}(F) \neq 2$

↳ counterexample: $\mathbb{Z}/_{\sim 2} = \{[0][1]\}$, check $[0] = -[0], [1] = -[1]$

↳ Corrolary: Let $v_1, \dots, v_p$ be linearly dependent. Then $\Gamma(v_1, \dots, v_p) = 0$

↳ proof: Fingerpractice

4) Any p-form Γ on a dim(V)-dimension vector space is the zero-map if $p > \dim(V)$

=> Terminology: Let V be a finite dimension vector space. Then an dim(V)-form is called a top form on that vector space.

=> Lemma: Let Ω and $\tilde{\Omega}$ be non-zero top forms on $(V, \oplus, \odot)$ with $\dim(V) < \infty$. Then there is an $c \in F^*$ such that $\tilde{\Omega} = c \diamond \Omega$.

↳ proof: Choose a basis $a_1, \dots, a_d$. Then the anti-symmetry on F => Componet $\underset{\Omega(e_{a_1}, \dots, e_{a_d})}{\underset{\shortparallel}{\Omega_{a_1, \dots, a_d}}} = \pm \underset{\Omega(e_1, \dots, e_d)}{\underset{\shortparallel}{\Omega_{1, \dots, d}}} = a \in F^*$

Analogously $\tilde{\Omega}_{a_1, \dots, a_d} = \pm \tilde{\Omega}_{1, \dots, d} = \tilde{a} \in F^*$

Thus $\tilde{\Omega}_{1, \dots, d} = \frac{\tilde{a}}{a} \Omega_{1, \dots, d}$ => claim

2. Determinant of an endomorphism

=> **Definition**: Let $\varphi: V \to V$ be an endomorphism on a d-dimension F-vector space $(V, \oplus, \odot)$

Choosing a basis $e_1, \dots, e_d$ for V and some non-zero top-form Ω on V, $\det \varphi := \frac{\Omega(\varphi(e_1), \dots, \varphi(e_d))}{\Omega(e_1, \dots, e_d)} \in F$ $\left\{\frac{e\text{-}\Omega(\dots)}{e\text{-}\tilde{\Omega}(\dots)}\right\}$

is the field element called the determinant of the endomorphism φ

↳ proof: (well-definedness)

1) independent of choice of Ω

↳ trivial because of the quotient

2) independent of choice of basis

↳ tutorial

=> Examples:

1) $\mathrm{id}_V : V \to V$, $v \mapsto v$ $\qquad \det \mathrm{id}_V = \frac{\Omega(\mathrm{id}(e_1), \dots, \mathrm{id}(e_d))}{\Omega(e_1, \dots, e_d)} \overset{\text{def id}}{\underset{\downarrow}{=}} \frac{\Omega(e_1, \dots, e_d)}{\Omega(e_1, \dots, e_d)} = 1_F$

↳ Comment: components of id with respect to basis:

$\mathrm{id}^a{}_b = \varepsilon^a(\mathrm{id}(e_b)) = \varepsilon^a(e_b) = \delta^a{}_b$

$\mathrm{id} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$ (the rest zero)

2) $f: V \to V$ endomorphism any F-vector space $(V, \oplus, \odot)$ $\mathrm{char}(F) \neq 2$

$$\det f = \frac{\Omega(f(e_1), f(e_2))}{\Omega(e_1, e_2)} \overset{\text{bin. con.}}{=} \frac{\Omega_{mn} f^m{}_1 f^n{}_2}{\Omega_{12}} = \frac{\Omega_{mn} f^m{}_1 f^1{}_2 + \Omega_{m2} f^m{}_1 f^2{}_2}{\Omega_{12}}$$

$$= \frac{\cancel{\Omega_{11}} f^1{}_1 f^1{}_2 + \Omega_{21} f^2{}_1 f^1{}_2 + \Omega_{12} f^1{}_1 f^2{}_2 + \cancel{\Omega_{22}} f^2{}_1 f^2{}_2}{\Omega_{12}}$$

$$= \frac{\Omega_{21} f^2{}_1 f^1{}_2 + \Omega_{12} f^1{}_1 f^2{}_2}{\Omega_{12}} = \frac{\Omega_{12} f^1{}_1 f^2{}_2 - \Omega_{12} f^2{}_1 f^1{}_2}{\Omega_{12}}$$

$$= f^1{}_1 f^2{}_2 - f^2{}_1 f^1{}_2 = ad - bc$$

$\dim V = 2$, $f^a{}_b = \epsilon^a(f(e_a)) \in F$

$$\begin{bmatrix} f^1{}_1 & \cdots & f^1{}_2 \\ \vdots & & \vdots \\ f^2{}_1 & \cdots & f^2{}_2 \end{bmatrix} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

↳ Remark: with lot's of extra mafia talk, one might write $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

=> Theorem: Let $\alpha: V \to V$, $\beta: V \to V$ be two endomorphisms then $\det(\alpha \circ \beta) = \det(\alpha) \cdot \det(\beta)$

↳ proof: β is an automorphism

note: $\tilde{e}_1, \dots, \tilde{e}_d$ linearly independent => Ω non-zero

1) $$\det(\alpha \circ \beta) \overset{\text{def}}{=} \frac{\Omega((\alpha \circ \beta)(e_1), \dots, (\alpha \circ \beta)(e_d))}{\Omega(e_1, \dots, e_d)}$$

$$\overset{\beta \text{ auto}}{=} \frac{\Omega(\alpha(\tilde{e}_1), \dots, \alpha(\tilde{e}_d))}{\Omega(\beta(e_1), \dots, \beta(e_d))} \cdot \frac{\Omega(\beta(e_1), \dots, \beta(e_d))}{\Omega(e_1, \dots, e_d)}$$

$$= \frac{\Omega(\alpha(\tilde{e}_1), \dots, \alpha(\tilde{e}_d))}{\Omega(\tilde{e}_1, \dots, \tilde{e}_d)} \cdot \frac{\Omega(\beta(e_1), \dots, \beta(e_d))}{\Omega(e_1, \dots, e_d)} = (\det \alpha) \cdot (\det \beta)$$

2) β is not invertible

Then $\beta(e_1), \dots, \beta(e_d)$ is not linearly independent

=> $\det \beta = \dfrac{\Omega(\beta(e_1), \dots, \beta(e_d))}{\Omega(e_1, \dots, e_d)} = 0$

Then also $\alpha \circ \beta$ is not invertible

=> $\det \alpha \circ \beta = 0$

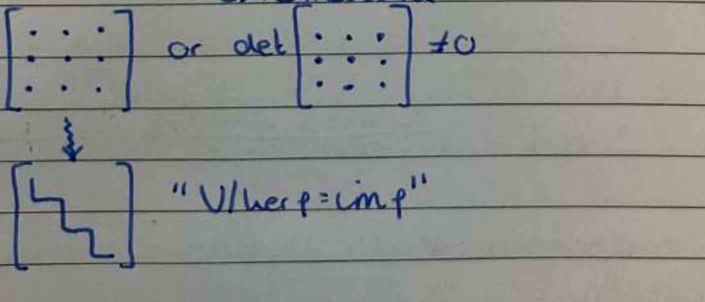
Therefore $\underbrace{\det(\alpha \circ \beta)}_{0_F} = \underbrace{\det(\alpha)}_{\in F} \cdot \underbrace{\det(\beta)}_{0_F}$

=> lemma: a) β not invertible => $\det \beta = 0$

↳ proof: See previous proof

b) $\det \beta = 0$ => β not invertible

↳ homework



=> lemma: $\det(\alpha) = \det(S \circ \alpha \circ S^{-1})$ for any endo α and auto S

↳ proof: $\det \neq 0$ $\det(S \circ \alpha \circ S^{-1}) = \det(S) \cdot \det(\alpha) \cdot \det(S^{-1})$

$= \det(S) \cdot \det(S^{-1}) \cdot \det(\alpha)$

$= \det(S^{-1} \circ S) \cdot \det(\alpha) = \det(\alpha)$

A comprehensive look at basis change et al

=> let $(V, \oplus, \odot)$ be a finite dimensional $\mathcal{F}$-vector space and
let $e_1, \ldots, e_d$ a basis for V and $\varepsilon^1, \ldots, \varepsilon^d$ the dual basis (wrt $e_1, \ldots, e_d$) for V^*
let $\tilde{e}_1, \ldots, \tilde{e}_d$ another basis for V and $\tilde{\varepsilon}^1, \ldots, \tilde{\varepsilon}^d$ the dual basis (wrt $\tilde{e}_1, \tilde{e}_d$) for V^*

Components of vectors, covectors, endomorphisms, automorphisms or generally (p,q)-tensors obtained with respect to one basis can be directly expressed in terms of another basis schematically

$$\begin{tikzcd} & T \arrow[dl, "{e,e}"'] \arrow[dr, "{\tilde{e},\tilde{e}}"] & \\ T^{\cdot\cdot}_{\cdots} \arrow[rr, "?"] & & T^{\cdot\cdot}_{\cdots} \end{tikzcd}$$

=> recall:

↳ A vector can be viewed as $(1,0)$-tensor T_v

↳ $v \in V$ define $T_v : V^* \to \mathcal{F}$

$$\sigma \mapsto \sigma(v)$$

Proof: additivety $T_v(\sigma + \tau) = (\sigma + \tau)(v) = \sigma(v) + \tau(v)$

Scalibility $T_v(\lambda\sigma) = (\lambda\sigma)(v) = \lambda \cdot \sigma(v)$

↳ A covector is a $(0,1)$-tensor

↳ $\sigma \in V^* \Rightarrow \sigma : V \to \mathcal{F}$ linear

$\Rightarrow \sigma$ is a $(0,1)$-tensor

↳ $f : V \to V$ can be viewed as a $(1,1)$-tensor

↳ $T_f : V^* \times V \to \mathcal{F}$

$$(\sigma, v) \mapsto T_f(\sigma, v) = \sigma(f(v))$$

↳ $b : V \times V \to \mathcal{F}$ is a $(0,2)$-tensor

↳ Special case: a $(0,0)$-tensor is an element of the field

=> **Definition**: $T = (p,q)$-tensor, $S = (r,s)$-tensor

$T \otimes S$ $(p+r, q+s)$-tensor

$(T \otimes S)(\sigma_{(1)}, \ldots, \sigma_{(p+r)}, v_{(1)}, \ldots, v_{(q+s)}) :=$

$T(\sigma_{(1)}, \ldots \sigma_{(p)}, v_{(1)}, \ldots, v_{(q)}) \cdot S(\sigma_{(p+1)}, \ldots \sigma_{(p+q)}, v_{(q+1)}, \ldots, v_{(q+s)})$

↳ Example: $v \otimes \sigma := T_v \otimes \sigma = (T_v \otimes \sigma)(\tau, \omega) := T_v(\tau) \cdot \sigma(\omega) = \tau(v) \cdot \sigma(\omega)$

=> **Definition**: The change of basis from e to $\tilde{e}$ can be encoded in an automorphism $S : V \to V$

(S Linear)

$e_a \mapsto S(e_a) := \tilde{e}_a$ for $a = 1, \ldots, d$

$\overset{!!✿}{}$ (normally: $v \mapsto S(v)$ for all $v \in V$)

✿ : $S(v) = S(v^a \cdot e_a) = v^a S(e_a) = v^a \tilde{e}_a$

↳ proof that S is an automorphism

injective) $\ker S := \{v \in V \mid S(v) = 0_V\}$

$S(v) = 0_V$

$\Rightarrow S(v^a e_a) = v^a S(e_a) = v^a \tilde{e}_a = 0$

$\Rightarrow v^1 = v^2 = \ldots = v^d = 0$ (since $\tilde{e}_a$ basis)

$\Rightarrow \ker S = \{0_V\}$

surjective) $\operatorname{im} S = V$

$$\begin{array}{l} \cong ? \\ V/\ker S \\ \cong ? \leftarrow \boxed{\ker S = \{0_V\}} \\ V \end{array}$$

Consider a change of basis given by $\tilde{e}_a := S(e_a)$ $a = 1, \ldots, d$ for some automorphism S. From this everything to do with basis changes follows (on finite dimensional vector spaces)

$\Rightarrow \varepsilon^a = (S^{-1})^*(\varepsilon^a)$

↳ proof: $((S^{-1})^*(\varepsilon^a))(\tilde{e}_b) \overset{\text{dual map}}{=} (\varepsilon^a \circ S^{-1})(\tilde{e}_b)$

$$\overset{\circ}{=} \varepsilon^a(S^{-1}(\tilde{e}_b))$$

$$\overset{\text{def } S^{-1}}{=} \varepsilon^a(e_b) = \delta^a{}_b$$

$$\Rightarrow (S^{-1})^*(\varepsilon^a) \underset{\uparrow}{=} \tilde{\varepsilon}^a$$

uniqueness dual basis

$\Rightarrow \tilde{e}_a = S^b{}_a e_b$ where $S^b{}_a := \varepsilon^b(S(e_a))$

↳ proof: $\tilde{e}_a = \underbrace{S(e_a)}_{\in V} = \underbrace{\varepsilon^b(S(e_a))}_{S^b{}_a} e_b$

$\Rightarrow \tilde{\varepsilon}^a = (S^{-1})^a{}_b \varepsilon^b$

↳ proof: $\tilde{\varepsilon}^a = (S^{-1})^*(\varepsilon^a) = ((S^{-1})^*(\varepsilon^a))(e_b)\varepsilon^b = (\varepsilon^a \circ S^{-1})(e_b)\varepsilon^b$

$= \underbrace{\varepsilon^a(S^{-1}(e_b))}_{(S^{-1})^a{}_b} \varepsilon^b$

Now turn to the induced change of components of tensors

$T^{a_1,\dots a_p}{}_{b_1,\dots,b_q} := T(\varepsilon^{a_1},\dots,e_{b_1})$

=> $\tilde{v}^a = (S^{-1})^a{}_b v^b \quad (\tilde{e}_a = S^b{}_a e_b)$

↳ plausibility check: $v^a e_a = v = \tilde{v}^a \tilde{e}_a$

$= \tilde{v}^a S^b{}_a e_b$

$= (S^{-1})^a{}_c v^c S^b{}_a e_b = S^b{}_a (S^{-1})^a{}_c v^c e_b$

$= \underbrace{(S \circ S^{-1})^b{}_c}_{\delta^b{}_c} v^c e_b = v^c e_c$

def of v

↳ proof: $\tilde{v}^a = \tilde{\varepsilon}^a(v) = ((S^{-1})^a{}_b \varepsilon^b)(v) \overset{\downarrow}{=} (S^{-1})^a{}_b v^b$

=> $\tilde{\sigma}_a = S^b{}_a \sigma_b$

↳ proof: $\tilde{\sigma}_a = \tilde{\sigma}_a(e_b) = (\tilde{\sigma}_a S^b{}_a)(e_b) = S^b{}_a \sigma_b$

invisible sums

=> $\tilde{T}^{a_1,\dots a_p}{}_{b_1,\dots,b_q} = (S^{-1})^{a_1}{}_{m_1} \dots (S^{-1})^{a_p}{}_{m_p} S^{n_1}{}_{b_1} \dots S^{n_q}{}_{b_q} \cdot T^{m_1,\dots,m_p}{}_{n_1,\dots,n_q}$

=> **Definition**: $f: V \to V$

$\operatorname{tr}(f) := f^a{}_a$ where $f^a{}_b := \varepsilon^a(f(e_b))$ wrt some basis e

↳ proof: well-definedness under change of basis

$\tilde{f}^a{}_a = (S^{-1})^a{}_m S^n{}_a f^m{}_n$

$= S^n{}_a \cdot (S^{-1})^a{}_m f^m{}_n = (S \circ S^{-1})^n{}_m f^m{}_n = \delta^n{}_m f^n{}_m = f^m{}_m$