

Exam Queueing Theory – April 15, 2025 (13.45 – 16.45)

This exam consists of six problems.  
Please put your name and student number on each sheet of paper.  
Using a simple (not graphic) scientific calculator is allowed.  
Motivate your answers.

1. In a repair shop of an airline company defective airplane engines are repaired. Defects occur according to a Poisson with a rate of 1 defective engine per 2 weeks. The mean repair time of an engine is  $2/3$  week. The repair time distribution can be well approximated by an Erlang-2 distribution. In the repair shop only one engine can be in repair at the same time.

(a)–4 pt Show that  $q_n$ , the probability that there are  $n$  engines in the repair shop, is given by

$$q_n = \frac{6}{5} \left(\frac{1}{4}\right)^n - \frac{8}{15} \left(\frac{1}{9}\right)^n.$$

(b)–2 pt Determine the mean sojourn time (waiting time plus repair time) of an engine in the repair shop.

The airline company has several spare engines in a depot. When a defect occurs, the defective engine is immediately replaced by a spare engine (if there is one available) and the defective engine is sent to the repair shop. After repair the engine is as good as new, and it is transported to the depot. When a defect occurs and no spare engine is available, the airplane has to stay on the ground and it has to wait till a spare engine becomes available.

(c)–2 pt Determine the minimal number of spare engines needed such that for 99% of the defects there is a spare engine available.

2. In a warehouse for small items orders arrive according to a Poisson stream with a rate of 6 orders per hour. An order is a list with the quantities of products requested by a customer. The orders are picked one at a time by one order picker. For a quarter of the orders the pick time is exponentially distributed with a mean of 10 minutes and for the other orders the pick time is exponentially distributed with a mean of 5 minutes.

(a)–2 pt Determine the Laplace-Stieltjes transform  $\tilde{B}(s)$  of the pick time in minutes of an arbitrary order.

(b)–4 pt Recall that the probability generating function of  $L$ , the limiting random variable of  $L(t)$ , is

$$P_L(z) = \frac{(1 - \rho)\tilde{B}(\lambda - \lambda z)(1 - z)}{\tilde{B}(\lambda - \lambda z) - z}. \quad (1)$$

Use (1) to show that the Laplace-Stieltjes transform of the sojourn time (waiting time plus pick time) in minutes of an arbitrary order is given by

$$\tilde{S}(s) = \frac{5}{32} \frac{3}{3 + 20s} + \frac{27}{32} \frac{1}{1 + 20s}.$$

[Hint: First determine a general expression for the LST  $\tilde{S}(s)$  of the sojourn time (waiting time plus pick time) in minutes of an arbitrary order from (1)]

(c)–2 pt Determine the fraction of orders for which the sojourn time is longer than half an hour.

(d)–1 pt Determine the mean sojourn time.

3. Consider the  $G/M/1$  queue. The interarrival times  $A$  have LST  $\tilde{A}(s)$  with mean  $1/\lambda$ , and the service times are exponentially distributed with mean  $1/\mu$ . Assume that  $\rho = \lambda/\mu < 1$ . Recall that the limiting probability that there are  $n$  customers in the system just before an arrival is  $a_n = (1 - \sigma)\sigma^n$ ,  $n = 0, 1, 2, \dots$ , with  $\sigma$  the unique root in  $(0, 1)$  of  $\sigma = \tilde{A}(\mu - \mu\sigma)$ .

Let the interarrival times with probability  $p_1 = 13/24$  have an exponential distribution with mean  $1/3$  minutes, and with probability  $p_2 = 11/24$  have an exponential distribution with mean  $1/2$  minutes. Let the service times have mean of  $1/6$  minutes.

(a)–2 pt Give an explicit expression for the distribution of the number of customers in the system just before an arrival.

(b)–3 pt Use mean value analysis to derive  $\mathbb{E}[L]$ , the mean number of customers in the system at an arbitrary time in equilibrium.

4. Consider an open network consisting of two queues. Queue 1 is a single server queue with service rate  $\mu_1$ . Queue 2 is an  $M/M/c/c$  queue with service rate  $\mu_2$ . Customers arrive to queue 1 following a Poisson process with rate  $\lambda$ . All customers that complete service in queue 1 route to queue 2. If queue 2 is non-saturated (contains less than  $c$  customers), then a customer arriving to queue 2 enters queue 2. However, if queue 2 is saturated, then a customer arriving to queue 2 is rejected and leaves the network. All customers that complete service at queue 2 leave the network.
- (a)–1 pt Give the states, the state space and the transition rates of the Markov chain that records the number of customers in the queues, and give the stability condition.
- (b)–4 pt Give the equilibrium distribution, and prove that this distribution is indeed the equilibrium distribution.
5. Jobs of a class  $c$ ,  $c = 1, \dots, C$ , arrive to a service center according to Poisson processes with rate  $\lambda(c)$ ,  $c = 1, \dots, C$ . All jobs have exponential service requirement with mean 1. Formulate the functions  $\kappa$ ,  $\gamma$ , and  $\delta$  of the  $(\kappa, \gamma, \delta)$  protocol for the following queues.
- (a)–2 pt The multi-server LIFO queue with preemptive resume policy: The service center has  $c$  servers. Each server may serve one job at a time. Jobs receive a time stamp upon arrival. If a server is vacant upon arrival of a job, then the job joins a free server. If all servers are busy upon arrival of a job, then the job will interrupt service of the job in service with the oldest time stamp. If a server completes service of a job, then the server selects the job with the oldest time stamp and will resume service for that job at the position where service was interrupted. Thus, the server will always serve the  $c$  jobs in the system with the oldest time stamps.
- (b)–2 pt The single-server LIFO queue with non-preemptive policy: Let the service center have 1 server, that may serve one job at a time. Jobs receive a time stamp upon arrival. If the server is vacant upon arrival of a job, then the job joins the free server. If the server is busy upon arrival of a job, then the arriving job will join the waiting room. If the server completes service, then the server will start serving the job in the waiting room with the oldest time stamp.

6. Let customers of types  $u = 1, \dots, U$  arrive to a network of  $J$  quasi-reversible queues according to a Poisson process with rate  $\mu_0(u)$ ,  $u = 1, \dots, U$ . A customer's type uniquely determines its route through the network along the sequence of queues  $r(u, 1), r(u, 2), \dots, r(u, L(u))$ . Let  $\{N_j(t)\}$  at state space  $S_j$  with transition rates  $q_j(\mathbf{c}_j, \mathbf{c}'_j)$ ,  $\mathbf{c}_j, \mathbf{c}'_j \in S_j$ , record the state of queue  $j$  and let  $\pi_j = (\pi_j(\mathbf{c}_j), \mathbf{c}_j \in S_j)$  denote the equilibrium distribution of  $\{N_j(t)\}$ ,  $j = 1, \dots, J$ .

- (a)–1 pt Give a complete description of the Markov chain that records the states of the network, i.e., state space, transition rates.
- (b)–1 pt Give the characterisation of quasi-reversibility for each of the queues.
- (c)–3 pt Show that the network itself is quasi-reversible.

Norm: Exam grade =  $1 + \text{total}/4$