Exam Queueing Theory — July 1, 2025 (13.45 – 16.45)

This exam consists of four problems, with a total of 36 points. Exam grade =(total points+4)/4. Please put your name and student number on each sheet of paper. Using a simple (not graphic) scientific calculator is allowed. Motivate all your answers.

1. At some registration office, customers arrive according to a Poisson process with rate 20 per hour. Each registration takes an exponentially distributed amount of time with mean 2 minutes. Customers will wait in order of arrival if needed. There is only a single registration officer, that can serve only one customer at a time.

   a. [**1 pt**] Give the traffic intensity ρ, and give two different interpretations for this quantity.

   b. [**2 pt**] Use Mean Value Analysis to determine the expected sojourn time of any (random) customer. Determine the expected number of customers in the system (including the customer at the desk, if present).

   c. [**2 pt**] Answer the same question if the registration officer immediately leaves his place if there are no more customers to help. He returns after a new customer has arrived, but this takes him an exponential time with mean 30 seconds.

   A new policy is adopted in which people can already fill out the forms in advance (at home). The officer gives priority to customers who have done this, which is for 20% of the customers. The mean service time of these customers is reduced to 1 minute. The other 80 % of customers still have a mean service time of 2 minutes. Assume the officer is always in place, and that no preemption takes place.

   d. [**2 pt**] Determine the expected sojourn time for the high priority customers.

2. Consider the $M/E_2/1$ queue with arrival rate λ. Let the phases of the E_2 distribution have rate μ. Let $\lambda = 1$, $\mu = 6$. Let q_i be the probability of i customers in the system, $i = 0, 1, 2, \ldots$.

   a. [**4 pt**] Assume that service is FIFO. Derive the distribution of q_i, $i = 0, 1, 2, \ldots$.

   b. [**4 pt**] Now assume that service is LIFO-PR. Derive the distribution of q_i, $i = 0, 1, 2, \ldots$.

3. Products arrive at a machine to receive service. Each service consists of two consecutive operations: the first takes an exponential amount of time with mean 15 seconds, the second operation takes an exponential amount of time with mean 20 seconds (thus, the server is alternating between type 1 and type 2 operations during a busy period). The arrivals occur as a Poisson process at rate 1 per minute, and products are served in order of arrival.

   a. [1 pt] Determine the LST (Laplace-Stieltjes Transform) of the total service time of a product (counting in minutes).

   b. [1 pt] Give Lindley's equation for the waiting time.

   c. [4 pt] Derive the Pollackzek-Khintchine formula for the LST of the distribution of the waiting time from Lindley's equation, i.e.,

$$\widetilde{W}(s) = \frac{(1-\rho)s}{s - \lambda + \lambda \widetilde{B}(s)}.$$

   [Hint: condition on the sojourn time and on the interarrival time.]

   d. [3 pt] Show that the LST of the waiting time of any product is given by

$$\tilde{W}(s) = \frac{5}{12} + \frac{15}{24}\frac{1}{s+1} - \frac{1}{24}\frac{5}{s+5}.$$

   e. [2 pt] Find the distribution function of the waiting time in stationarity.

   f. [2 pt] Determine the expected sojourn time.

4. Consider a tandem network of 2 single-server FIFO queues. Jobs of type t arrive at queue 1 according to a Poisson process with rate λ_t, $t = 1, \ldots, T$. The service times of jobs of type t in queue i are independent and identically distributed and have an exponential distribution with rate $\mu_{it} = \mu_i$, $i = 1, 2$, $t = 1, \ldots, T$. Jobs that have been served in queue 1 route to queue 2. Customers that complete service in queue 2 leave the network. Both queues have an unlimited waiting room.

   a. [2 pt] Model this system as a network of queues with multiple customer types and give the state space and the transition rates of the Markov chain that records the numbers and types of the customers in each position in the queues.

   b. [1 pt] Give the stability condition for this network. Why is the condition $\mu_{it} = \mu_i$, $t = 1, \ldots, T$ imposed on queue i, $i = 1, 2$?

   c. [2 pt] Give the equilibrium distribution of the Markov chain that records the type of the customers in each position in the queues. Motivate your answer.

   d. [3 pt] Now assume that the service time distribution in queue 2 is changed from exponential to deterministic with the same mean. Give an explicit expression for the LST of the sojourn time of a customer in the network. Motivate your answer.