Examination: Continuous Optimization

3TU- and LNMB-course, Utrecht, January 14, 2013, 13.30-16.30

Ex. 1

(a) Let $0 \neq a \in \mathbb{R}^n$ be given. Show that the matrix $A := aa^T$ is positive semidefinite and has rank 1.

(b) For a symmetric (n×n)-matrix C show:

C is positive definite $\iff$ $C \bullet A > 0$ holds for all positive semidefinite matrices $A \neq 0$

(Here, for $C = (c_{ij})$, $A = (a_{ij})$, $C \bullet A$ denotes the inner product $C \bullet A = \sum_{i,j} c_{ij} \cdot a_{ij}$.)

Ex. 2 Let $f : \mathbb{R}^m \to \mathbb{R}$ be a convex function $f(y)$ on $\mathbb{R}^m$ and let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ be given.

(a) Show that the function $g(x) := f(Ax + b)$ is a convex function of x on $\mathbb{R}^n$.

(b) Suppose that f is strictly convex. Show that then $g(x) := f(Ax + b)$ is strictly convex if and only if A has (full) rank n.

Hint: Recall that f is strictly convex if for any $y_1 \neq y_2$, $0 < \lambda < 1$ it holds: $f(\lambda y_1 + (1-\lambda)y_2) < \lambda f(y_1) + (1-\lambda) f(y_2)$.

Ex. 3 We consider the convex program:

$$(CO) \quad \min_{x \in \mathbb{R}^2} e^{-x_2} \quad \text{s.t.} \quad \sqrt{x_1^2 + x_2^2} - x_1 \leq 0$$

(a) Show that (CO) is a convex program. Show that the feasible set $\mathcal{F}$ is given by $\mathcal{F} = \{(x_1, x_2) \mid x_1 \geq 0, x_2 = 0\}$ and determine (all) minimizers of (CO).

(b) Analyse the Wolfe dual (WD) and the Lagrangian dual (D) of (CO) and show that for the corresponding optimal values we have: $v(WD) = -\infty < v(D) = 0 < v(CO) = 1$. *(Hint for analyzing (D): Show that for any fixed x_2 it follows:* $\inf_{x_1 \in \mathbb{R}} \sqrt{x_1^2 + x_2^2} - x_1 = 0$.)

Ex. 4

(a) Consider with $0 \neq c \in \mathbb{R}^2$ the program:

$$(P_0) : \quad \max_{x \in \mathbb{R}^2} c^T x \quad \text{st.} \quad x \in \mathcal{F} := \{(x_1, x_2) \mid x_1^2 + x_2^2 \leq 1\}.$$

Give the extreme points of the feasible set $\mathcal{F}$ and determine the maximizer of (P_0). Sketch the problem (feasible set, maximizer, level set of $c^T x$) and indicate that every point $x \in \mathcal{F}$ can be written as a convex combination of extreme points of $\mathcal{F}$.

(b) Let $f : \mathcal{F} \to \mathbb{R}$ be convex on a given compact convex set $\mathcal{F} \subset \mathbb{R}^n$ and consider the problem:

$$(P) : \quad \max f(x) \quad \text{st.} \quad x \in \mathcal{F}.$$

Show that the maximum value of (P) is always (also) attained at some extreme point of $\mathcal{F}$.

Hint: Use the Krein-Milman theorem and Jensen's inequality.

Ex. 5
Consider the unconstrained minimization problem $\min_{x\in\mathbb{R}^n} \; q(x)$ on $\mathbb{R}^n$ with the quadratic function $q(x) := \frac{1}{2}x^T Ax + b^T x$, where A is a positive definite n×n-matrix.

Determine the (global) minimizer $\overline{x}$ of q on $\mathbb{R}^n$.
For given point $x_0 \neq \overline{x}$, show that the "Newton direction" $d := -[\nabla^2 q(x_0)]^{-1}\nabla q(x_0)$ is a descent direction (for q in x_0).
Show furthermore that the Newton iteration step $x_0 \to x_1$ (for minimizing q) yields the minimizer $\overline{x}$ (in one step).
Hint: Recall that A is positive definite if and only if A^{-1} is positive definite.

Ex. 6 Consider the constrained minimization problem:

$$(P): \qquad \min_{x\in\mathbb{R}^2} \; x_1 + x_2 \qquad \text{s.t.} \qquad -x_1^2 - x_2 \leq 0 \quad \text{and} \quad -x_1 \leq 0\,.$$

(a) Compute all KKT-points of (P).

(b) Show that $\overline{x} := 0$ is a strict local minimizer of order 1.
Is $\overline{x}$ the global minimizer? (explain)

Points: 36+4 =40

1	a	:	2	2	a	:	3	3	a	:	3	4	a	:	3	5	:	4	6	a	:	3
	b	:	4		b	:	4		b	:	4		b	:	3					b	:	3

A copy of the lecture-sheets may be used during the examination.
Good luck!