Examination: Continuous Optimization

3TU- and LNMB-course, Utrecht January 16, 2012, 16.00-19.00

Ex. 1 Recall that the *affine hull* of a set $S \subset \mathbb{R}^n$ is defined by:

$$\text{aff}\,(S) = \{x \mid x = \sum_{i=1}^{k} \lambda_i x^i,\ x^i \in S,\ \lambda_i \in \mathbb{R},\ \sum_{i=1}^{k} \lambda_i = 1,\ k \geq 1\}$$

Consider a non-empty subset $S \subset \mathbb{R}^n$. Show that the set aff (S) is an affine space, *i.e.* aff (S) has the form aff $(S) = \overline{x} + V$ with $\overline{x} \in S$ and V a linear subspace of $\mathbb{R}^n$.

Ex. 2 Consider the convex program

$$(CO) \qquad \min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad g_j(x) \leq 0,\ j = 1, \ldots, m\ ,$$

with convex functions $f, g_j \in C^1(\mathbb{R}^n, \mathbb{R})$.
Show that $(\overline{x}, \overline{y})$ is a saddle point for the Lagrangian function $L(x, y)$ of (CO) if and only if $\overline{x}$ is feasible for (CO) and satisfies the KKT-conditions (Karush-Kuhn-Tucker conditions) with a multiplier vector $\overline{y} \geq 0$.

Ex. 3 Recall that in the set S_n of symmetric $n \times n$-matrices, $A \bullet C$ denotes the "inner product", $A \bullet C = \sum_{i,j} a_{ij} c_{ij}$, $A = (a_{ij}), C = (c_{ij}) \in S_n$.
Consider the set of positive semidefinite $n \times n$-matrices $S_n^+ := \{A \in S_n \mid x^T A x \geq 0\ \forall x \in \mathbb{R}^n\}$.

(a) Show that S_n^+ is a convex cone.

(b) Verify the identity $x^T A x = A \bullet x x^T$ for $A \in S_n, x \in \mathbb{R}^n$.

(c) Consider the dual cone of S_n^+, $(S_n^+)^* := \{B \in S_n \mid B \bullet A \geq 0\ \forall A \in S_n^+\}$.
Show that the relation holds: $S_n^+ = (S_n^+)^*$ (*i.e.*, S_n^+ is self-dual.)

Ex. 4

(a) Let the feasible set be defined by $\mathcal{F} = \{x \in \mathbb{R}^n \mid g_j(x) \leq 0, j = 1, \ldots, m\}$, where the functions $g_j(x)$, $j = 1, \ldots, m$, are convex on $\mathbb{R}^n$. Show that $\mathcal{F}$ is a closed, convex set.

(b) Is also the following converse true? For any $g : \mathbb{R}^n \to \mathbb{R}$ it holds :

$$\mathcal{F} := \{x \in \mathbb{R}^n \mid g(x) \leq 0\} \text{ is convex } \Rightarrow g \text{ is convex on } \mathcal{F}$$

Hint: Prove it or find a counterexample.

Ex. 5 Apply the conjugate gradient method (see lecturesheets, Th.11.3) to a quadratic function $q(x) = \frac{1}{2}x^T Ax + b^T x$ (A positive definite). (Recall $g_j := \nabla q(x_j)$; d_j are the search directions).

(a) Show that the following recursion is true: $g_{j+1} = g_j + t_j A d_j$.

(b) If we start with the direction $d_0 = -g_0$, then the following relations are true for any k, $0 \le k \le n-1$:
(1) $g_j^T g_i = 0 \;\; \forall i, j,\; 0 \le i < j \le k$ and (2) span $\{d_0, \dots, d_k\} =$ span $\{g_0, \dots, g_k\}$.
Hint: Induction wrt. k; You may use all facts proven for the method of conjugate directions in the course CO.
Recall: span $\{d_0, \dots, d_k\}$ *denotes the linear space spanned by* $d_0, \dots, d_k$.

Ex. 6 Consider the constrained minimization problem:

$$(P) \quad \min\ x_2 \ \text{ s.t. } \ g_1(x) := -x_1 \le 0, \ \ g_2(x) := x_1 - x_2^3 \le 0 \,.$$

(a) Show that at the point $\overline{x} = (0,0)$ the MFCQ constraint qualification is not satisfied. Show further that $\overline{x}$ is not a KKT point.

(b) Show that $\overline{x} = (0,0)$ is the unique solution of (P). (It is even a strict local minimizer of order $p = 1$).
Hint: Show this by direct verification.

Ex. 7 Given the program:

$$(P) \quad \min\ x^2 \ \text{ s.t. } \ g(x) := 1 - x \le 0 \,.$$

We consider the penalty problem P_r : $\min p_r(x) := x^2 + r(g^+(x))^2$
and the exact penalty problem $\hat{P}_r$: $\min \hat{p}_r(x) := x^2 + r g^+(x)$. (Here $g^+(x) := \max\{0, g(x)\}$.)

(a) Compute the solution $\overline{x}$ of (P) with corresponding Lagrange multiplier (solve the system of KKT-conditions).

(b) Compute the solutions x_r of P_r and the solutions $\hat{x}_r$ of $\hat{P}_r$ for $r > 0$.

(c) Analyse the error $|x_r - \overline{x}|$ and $|\hat{x}_r - \overline{x}|$ for $r \to \infty$.

Points: **38+4 =42** [mark: points/10 with a max of 10+]

1 : 3	2 : 5	3 a : 2	4 a : 3	5 a : 2	6 a : 3	7 a : 2
		3 b : 2	4 b : 2	5 b : 3	6 b : 3	7 b : 4
		3 c : 3				7 c : 1

A copy of the lecture-sheets can be used during the examination.
Good luck!