Main Exam: Continuous Optimization

You may assume the result of a (sub)question in answering the (sub)questions coming after it. You may use any statement from the homeworks or lecture slides in your answers. Workings must be shown. Please return the question sheet after the exam. Good Luck!

1. (a) Prove that the function $f(x) = \begin{cases} 2x : x \geq 0 \\ x : x \leq 0 \end{cases}$ is convex. [5 points]

   (b) For f as above, compute $\partial f(0)$ (the subgradient set at 0). [5 points]

   (c) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a strictly convex function. Assume that f attains its minimum value at $\mathbf{x}^* \in \text{dom}(f)$. Prove that $\mathbf{x}^*$ is the *unique* minimizer. [5 points]

2. Let $f(x_1, x_2) = \frac{x_1^4}{e^{x_1^2+x_2}}$.

   (a) Express f via a computation graph, using the elementary functions $(\cdot)^2$, $e^{(\cdot)}$, $\frac{1}{(\cdot)}$, $\cdot + \cdot$, $\cdot \times \cdot$ (do not use $(\cdot)^4$ as an elementary function). [5 points]

   (b) Show the forward phase computation for $\nabla f(1, -1)$. [5 points]

   (c) Show the backward phase computation for $\nabla f(1, -1)$. [5 points]

3. Let $\mathbf{y}_1, \ldots, \mathbf{y}_m \in \mathbb{R}^n$ with $\|\mathbf{y}_i\|_2 \leq 1$, $\forall i \in [m]$. Examine the optimization problem:

$$\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}), \text{ where } f(\mathbf{x}) := \frac{1}{2m} \sum_{i=1}^{m} \|\mathbf{x} - \mathbf{y}_i\|_2^2.$$

   (a) Prove that $f(\mathbf{x})$ is 1-smooth and 1-strongly convex. [5 points]

   (b) Prove that $\mathbf{x}^* = \sum_{i=1}^m \mathbf{y}_i / m$ is the optimal solution. [5 points]

   (c) Prove that for any starting point $\mathbf{x}_0 \in \mathbb{R}^n$, one step of gradient descent with exact line search satisfies $\mathbf{x}_1 = \mathbf{x}^*$. [5 points]

   (d) Add the constraint $\mathbf{x} \in \mathcal{C} := [-1, 1]^n$ to the above optimization problem and assume that $\sum_{i=1}^m \mathbf{y}_i / m = (2, \ldots, 2)^\mathsf{T} \in \mathbb{R}^n$ (the vector with all coordinates equal to 2). Starting from the origin $\mathbf{x}_0 := \mathbf{0}_n$, what is the next iterate $\mathbf{x}_1$ computed by projected gradient descent using a step size of 1 (here we project onto $\mathcal{C}$)? [5 points]

4. Consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & f_0(\mathbf{x}) := \sum_{i=1}^{n} u_i(x_i) \\ \text{subject to} \quad & f_i(\mathbf{x}) := -x_i \leq 0, \; i = 1, \ldots, n, \\ & f_{n+1}(\mathbf{x}) := \sum_{i=1}^{n} x_i - (n+2) \leq 0 \end{aligned}$$

where the functions $u_i \colon \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$, are convex and twice continuously differentiable. Let $\mathbf{x}^*$ denote a global minimizer to the above problem.

(a) Prove that $\mathbf{x}^*$ is a KKT point. [5 points]

(b) Use the KKT conditions to show that the value of $u_i'(x_i^*)$ is the same for each i with $x_i^* \neq 0$. Here u_i' is the derivative of u_i. [10 points]

(c) Suppose we define the functions $u_i(t) := t$ (used to the define the objective f_0 above), $\forall i \in [n], t \in \mathbb{R}$. Using this choice of objective, compute the logarithmic barrier central path point $\mathbf{x}^*(1/2)$. [5 points]

5. Examine the following semidefinite optimization problem (P5):

$$\min \; x_1$$
$$\begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{pmatrix} \succeq 0 \qquad \text{(P5)}$$

Let $\mathbf{x} = (x_1, x_2)^\mathsf{T}$. For simplicity of notation, you may use $M[\mathbf{x}]$ to denote the matrix on the left hand side of (P5).

(a) Prove that $\mathbf{x}$ is feasible for (P5) iff $\|\mathbf{x}\|_2 \leq 1$. [5 points]

(b) Compute an optimal solution to (P5). [5 points]

(c) Formulate the dual semidefinite program (D5). [5 points]

(d) Compute an optimal solution to (D5). [5 points]
(Hint: find a solution having the same value as part b)

6. (Automatic additional points) [10 points]

Question:	1	2	3	4	5	6	Total
Points:	15	15	20	20	20	10	100