Practice Exam: Continuous Optimization

1. Consider the problem $\min_x\{x^2 : x \geq 1\}$. For a parameter $\rho > 0$, this problem can be approximated by the unconstrained optimization problem [3 points]

$$\begin{aligned} \min_x \quad & x^2 - \rho \ln(x-1) \\ \text{s.t.} \quad & x > 1. \end{aligned} \tag{A}$$

Find the optimal solutions to (A) as a function of $\rho > 0$, and find the limit of these optimal solutions as $\rho \to 0^+$.

2. Consider a closed nonempty set $\mathcal{C} \subseteq \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}$ defined to be the distance to the set for some given norm, i.e. [3 points]

$$f(\mathbf{x}) = \min_{\mathbf{y}}\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{y} \in \mathcal{C}\}.$$

Prove that if $\mathcal{C}$ is a convex set then f is a convex function.
[You may assume that the minimum defining f is attained.]

3. For a fixed parameter $\alpha \in \mathbb{R}$, consider the function $f_\alpha(\mathbf{x}) = \exp(x_1+x_2)+\alpha x_1^2+x_2^4$.
   (a) For what values of the parameter $\alpha \in \mathbb{R}$ is f_α a convex function? [3 points]

   From now on consider having $\alpha = 1$ (for which we have that f_α is a convex function).

   (b) By considering the function at $\mathbf{x} = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, show that $f_1(\mathbf{y}) \geq 1 + y_1 + y_2$ for all $\mathbf{y} \in \mathbb{R}^2$. [2 points]
   (c) Give the direction of steepest descent of f_1 at $\mathbf{x} = \mathbf{0}$. [1 point]
   (d) Give the Newton direction of f_1 at $\mathbf{x} = \mathbf{0}$. [2 points]

   [These directions do not need to be normalised.]

4. Consider the problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & 4x_1 + x_2^2 \\ \text{s.t.} \quad & x_2 \geq x_1^2 \\ & \mathbf{x} \in \mathbb{R}^2. \end{aligned} \tag{B}$$

   (a) Show that problem (B) is a convex problem. [2 points]
   (b) Does Slater's condition hold for problem (B)? (You must justify your answer.) [1 point]
   (c) Find the KKT point(s) for problem (B). [3 points]
   (d) What is the global minimizer for problem (B), and prove that this minimizer is a local minimizer of order 2. [3 points]
   (e) Formulate and solve the Langrangian Dual problem to problem (B). [4 points]

5. Let $M(x,y) := \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix}$ for $x, y \in \mathbb{R}$ and let $\lambda(M(x,y)) := \max\{|\lambda_1(M(x,y))|, |\lambda_2(M(x,y)|\}$ be the absolute value of the eigen value of $M(x,y)$ of largest absolute value.

(a) Formulate a semidefinite program that solves the problem of finding $x, y \in \mathbb{R}$ minimizing $\lambda(M(x, y))$. [1 point]

(b) Formulate the corresponding dual semidefinite program. [3 points]

(c) Show that $x = y = 0$ is the optimal solution by exhibiting a dual solution whose value is equal to $\lambda(M(0, 0))$. [2 points]

6. Let $\mathbf{y}_1, \ldots, \mathbf{y}_N \in \mathbb{R}^n$. Examine the optimization problem:

$$\min_{\mathbf{x} \in \mathbb{R}^n} \sum_{i=1}^{N} \|\mathbf{x} - \mathbf{y}_i\|_2^2$$

(a) Prove that $\mathbf{x}^* = \sum_{i=1}^{N} \mathbf{y}_i / N$ is the optimal solution. [2 points]

(b) Show that $\mathbf{x}^*$ is a local minimum of order 2. [1 point]

7. (Automatic additional points) [4 points]

Question:	1	2	3	4	5	6	7	Total
Points:	3	3	8	13	6	3	4	40

A copy of the lecture-sheets may be used during the examination. You may use any results from the lecture slides in your answers (Lemmas, Theorems, Corollaries, Exercises, etc.), however you should reference the result.

Hints:

1. $$\begin{pmatrix} a & b \\ b & c \end{pmatrix}^{-1} = \frac{1}{ac - b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$$
2. $\begin{pmatrix} a & b \\ b & c \end{pmatrix} \succeq 0$ *iff* $a, c \geq 0$ *and* $ac \geq b^2$.
3. *A norm* $\| \bullet \|$ *on* $\mathbb{R}^n$ *has the following properties:*
   (a) $\|\lambda \mathbf{x}\| = |\lambda| \, \|\mathbf{x}\|$ *for all* $\lambda \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$;
   (b) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$;
   (c) $\|\mathbf{x}\| > 0$ *for all* $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.