Retake: Continuous Optimisation

13:30 – 16:30, Monday 19th February 2018

Hints at the end of the paper. Workings must be shown. Good Luck!

1. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a convex function, and let $\mathsf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then show that $g(\mathbf{x}) := f(\mathsf{A}\mathbf{x} + \mathbf{b})$ is a convex function on $\mathbb{R}^n$. [3 points]

2. For a fixed parameter $\alpha \in \mathbb{R}$, consider the function $f_\alpha : \mathbb{R}^2 \to \mathbb{R}$ given by $f_\alpha(x_1, x_2) = 2x_1^2 - 2x_1x_2 + x_2^2 + \alpha \sin x_1$.

   (a) For what values of the parameter $\alpha \in \mathbb{R}$ is f_α a convex function on $\mathbb{R}^2$? [3 points]

   From now on we will fix $\alpha = 1$, i.e. $f_1(x_1, x_2) = 2x_1^2 - 2x_1x_2 + x_2^2 + \sin x_1$ (noting that f_1 is convex), and we will consider the function f_1 at $\widehat{\mathbf{x}} = \begin{pmatrix}0, & 1\end{pmatrix}^\mathsf{T}$.

   (b) What is the gradient vector and Hessian matrix for f_1 at $\mathbf{x} = \widehat{\mathbf{x}}$? [1 point]

   (c) By considering f_1 at $\mathbf{x} = \widehat{\mathbf{x}}$, show that $f_1(\mathbf{y}) \geq 2y_2 - y_1 - 1$ for all $\mathbf{y} \in \mathbb{R}^2$. [1 point]

   (d) Find the steepest descent direction of f_1 from $\widehat{\mathbf{x}}$. [1 point]

   (e) Find the Newton direction of f_1 from $\widehat{\mathbf{x}}$. [2 points]

   [The directions from parts (d) *and* (e) *need not be normalised, i.e. need not have a length equal to 1.]*

3. Consider the convex optimisation problem [3 points]

$$\min_{\mathbf{x}\in\mathbb{R}^n}\{f(\mathbf{x}) : g_j(\mathbf{x}) \leq 0 \text{ for all } j = 1, \ldots, m\}, \qquad \text{(CO)}$$

   with convex functions $f, g_j \in C^1(\mathbb{R}^n, \mathbb{R})$.
   Show that if $\overline{\mathbf{x}} \in \mathcal{F}$ is a feasible point for this problem which satisfies the KKT-conditions (Karush-Kuhn-Tucker conditions) with a multiplier vector $\overline{\mathbf{y}} \in \mathbb{R}^m_+$ then $(\overline{\mathbf{x}}, \overline{\mathbf{y}})$ is a saddle point for the Lagrangian function $L(\mathbf{x}, \mathbf{y})$ of (CO).

4. Consider the following minimisation problem:

$$\begin{aligned} \min_{\mathbf{x}} \quad & 3x_1 + x_2 \\ \text{s. t.} \quad & x_1^2 + x_2^2 \leq 1 \\ & x_1^2 \leq 1 + x_2 \\ & \mathbf{x} \in \mathbb{R}^2. \end{aligned}$$

   We will first find the global minimiser(s) for this problem:

   (a) Is this problem a convex optimisation problem? [2 points]

   (b) Find the KKT points for this problem. [4 points]

   (c) Find a global minimiser for this problem. [2 points]

   We will now construct the Langrangian dual problem to this minimisation problem:

   (d) What is the Lagrangian dual problem to this problem? [4 points]

5. In this question you may NOT use Lemma 7.3, the result of exercise 7.4, the result of exercise 7.5, the example directly after Definition 7.9, nor the example directly before Lemma 7.13.

   Consider a norm $p : \mathbb{R}^n \to \mathbb{R}$ and the set

$$\mathcal{K} = \{(x_0, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n : p(\mathbf{x}) \leq x_0\}$$

   (a) Prove that $\mathcal{K}$ is a proper cone in $\mathbb{R} \times \mathbb{R}^n$. [4 points]
   (You may assume that $\mathcal{K}$ is a closed set.)

   (b) Letting $q : \mathbb{R}^n \to \mathbb{R}$ be the dual norm to p, defined by [3 points]

$$q(\mathbf{y}) := \sup\{\mathbf{x}^\mathsf{T}\mathbf{y} : \mathbf{x} \in \mathbb{R}^n,\, p(\mathbf{x}) \leq 1\},$$

   and considering the standard inner product $\langle (y_0, \mathbf{y}), (x_0, \mathbf{x}) \rangle = y_0 x_0 + \mathbf{y}^\mathsf{T}\mathbf{x}$, find a characterisation for the dual cone to $\mathcal{K}$ in terms of q.

   (c) Suppose we are given the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n$. Then using the norm p as a measure of distance (i.e. the distance between $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is given by $p(\mathbf{x} - \mathbf{y})$), formulate as a conic optimisation problem the problem of finding an $\mathbf{x} \in \mathbb{R}^n$ which minimises the maximum distance between $\mathbf{x}$ and the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$. [3 points]

6. (Automatic additional points) [4 points]

Question:	1	2	3	4	5	6	Total
Points:	3	8	3	12	10	4	40

A copy of the lecture-sheets may be used during the examination.
You may use any results from the lecture slides in your answers (Lemmas, Theorems, Corollaries, Exercises, etc.), unless otherwise stated in the question. If possible you should reference the results used.

Hints: 1. $\begin{pmatrix} a & b \\ b & c \end{pmatrix}^{-1} = \frac{1}{ac - b^2}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$;

2. $\begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathcal{PSD}^2$ *if and only if* $a + c \geq 0$ *and* $ac - b^2 \geq 0$;

3. $\sin 0 = 0$ *and* $\cos 0 = 1$;

4. *A function* $p : \mathbb{R}^n \to \mathbb{R}$ *is a norm if:*
   i. $p(\lambda\mathbf{x}) = |\lambda|\, p(\mathbf{x})$ *for all* $\lambda \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$ *(absolutely homogeneous),*
   ii. $p(\mathbf{x} + \mathbf{y}) \leq p(\mathbf{x}) + p(\mathbf{y})$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *(triangle inequality),*
   iii. $p(\mathbf{x}) > 0$ *for all* $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ *(positive definite).*