Exam: Continuous Optimisation

13:30 – 16:30, Monday 15^{th} January 2018

Hints at the end of the paper. Workings must be shown. Good Luck!

1. (a) Show that if $f_1, f_2 : \mathcal{C} \to \mathbb{R}$ are both convex functions then so is the function $f : \mathcal{C} \to \mathbb{R}$ given by $f(\mathbf{x}) = \max\{f_1(\mathbf{x}), f_2(\mathbf{x})\}$. [2 points]

   (b) Is the following function $f : \mathbb{R} \to \mathbb{R}$ a convex function? Justify your answer. [1 point]

$$f(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ \exp(x) & \text{if } x \geq 0. \end{cases}$$

2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x_1, x_2) = \sin(x_1) + x_1^2 + 2x_1x_2 + 3x_2^2$.

   (a) What is the gradient vector and Hessian matrix for this function? [1 point]

   (b) Is f a convex function over $\mathbb{R}^2$? Justify your answer. [2 points]

   (c) Find the steepest descent direction of f from $\widehat{\mathbf{x}} = \begin{pmatrix}0, & 1\end{pmatrix}^\mathsf{T}$. [1 point]

   (d) Find the Newton direction of f from $\widehat{\mathbf{x}} = \begin{pmatrix}0, & 1\end{pmatrix}^\mathsf{T}$. [2 points]

   [These directions need not be normalised, i.e. need not have a length equal to 1.]

3. In this question we will prove that for convex optimisation problems, KKT points are global minimisers. In this question you are allowed to use definitions from any of the chapters, but you may only use results (i.e. Corollaries, Theorems, Lemmas and Exercises) from Chapter 1.

   For convex differentiable functions $f, g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$, let

$$\mathcal{F} = \{\mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \leq 0 \text{ for all } i \in \{1, \ldots, m\}\}$$

   and consider the problem of minimising the function f over $\mathcal{F}$.

   Let $\mathbf{x}_0 \in \mathcal{F}$ be a KKT point for this with corresponding multipliers $\boldsymbol{\lambda} \in \mathbb{R}^m_+$.

   (a) Show that for all $\mathbf{z} \in \mathcal{F}$ and all $i \in \mathcal{J}_{\mathbf{x}_0}$ we have $\nabla g_i(\mathbf{x}_0)^\mathsf{T}(\mathbf{z} - \mathbf{x}_0) \leq 0$. [2 points]

   (b) Show that for all $\mathbf{z} \in \mathcal{F}$ we have $\left(\sum_{i=1}^m \lambda_i \nabla g_i(\mathbf{x}_0)\right)^\mathsf{T}(\mathbf{z} - \mathbf{x}_0) \leq 0$. [2 points]

   (c) Show that for all $\mathbf{z} \in \mathcal{F}$ we have $f(\mathbf{z}) \geq f(\mathbf{x}_0)$. [1 point]

P.T.O.

4. Consider the following minimisation problem:

$$\begin{aligned} \min_{\mathbf{x}} \quad & x_1^2 - 2x_1 - x_2^2 \\ \text{s. t.} \quad & 2x_1^2 \leq 2 - x_2^2. \end{aligned}$$

We will first find the global minimiser(s) for this problem:

(a) Show that the linear independency constraint qualification holds at all feasible points of this problem. [2 points]

(b) Find the KKT points for this problem. [4 points]

(c) Find the global minimiser(s) for this problem. [2 points]

We will now construct the Langrangian dual problem to this minimisation problem:

(d) What is the Lagrangian dual problem to this problem? [4 points]

5. Consider the following set in $\mathbb{R}^3$:

$$\mathcal{K} = \left\{ \mathbf{x} \in \mathbb{R}^3 : \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_1 & 0 \\ x_3 & 0 & x_1 \end{pmatrix} \in \mathcal{PSD}^3 \right\}$$

You may assume that $\mathcal{K}$ is a closed set and that $\mathcal{PSD}^3$ is a proper cone.

(a) Show that $\mathcal{K}$ is a closed convex cone. [2 points]

(b) Show that $\mathcal{K}$ is full-dimensional. [2 points]

(c) Show that $\mathcal{K}$ is a proper cone. [2 points]

(d) Consider three random variables, Y_1, Y_2, Y_3 and suppose that $\text{corr}(Y_2, Y_3) = 0$ and $\text{corr}(Y_1, Y_3) = 0.5$. Formulate as a conic optimisation problem over $\mathcal{K}$, the problem of finding the minimum possible value of $\text{corr}(Y_1, Y_2)$. [3 points]

(e) Given that $\mathcal{K}$ is a self dual cone, i.e. $\mathcal{K}^* = \mathcal{K}$, formulate the conic dual problem to the optimisation problem from part (d). [1 point]

6. (Automatic additional points) [4 points]

Question:	1	2	3	4	5	6	Total
Points:	3	6	5	12	10	4	40

A copy of the lecture-sheets may be used during the examination.
You may use any results from the lecture slides in your answers (Lemmas, Theorems, Corollaries, Exercises, etc.), unless otherwise stated in the question. If possible you should reference the results used.

Hints: 1. $\begin{pmatrix} a & b \\ b & c \end{pmatrix}^{-1} = \dfrac{1}{ac - b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix};$

2. $\begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathcal{PSD}^2$ *if and only if* $a + c \geq 0$ *and* $ac - b^2 \geq 0$.