Ex $f : F \to \mathbb{R}$, $F \subset \mathbb{R}^n$, F convex, $f \in C^1$, f convex.

Consider: (P): $\min f(x)$ s.t. $x \in F$

Then

$$\bar{x} \text{ (global) min} \iff \nabla f(\bar{x})^T (x - \bar{x}) \geq 0 \quad \forall x \in F$$

$F = \mathbb{R}^n$
$\bar{x}$ min
$\Rightarrow \nabla f = 0$
$\Leftarrow$ convex
$\Rightarrow$ nv

Pf. "$\Leftarrow$": Since $f \in C^1$ is convex:

$$f(x) \geq \underbrace{f(\bar{x})}_{\geq f(\bar{x})} + \underbrace{\nabla f(\bar{x})^T (x - \bar{x})}_{\geq 0} \quad \forall x \in F$$

i.e., $\bar{x}$ is min.

"$\Rightarrow$": Let $x \in F$ (arbitrarily), put $x_\lambda := \bar{x} + \lambda(x - \bar{x}) \in F$

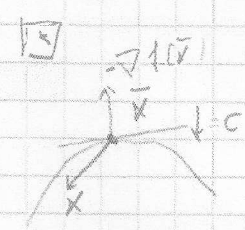
$\lambda \in [0,1]$. Since $f \in C^1$, for $\lambda > 0$:

$$0 \leq f(x_\lambda) - f(\bar{x}) = \lambda \nabla f(\bar{x})^T (x - \bar{x}) + o(|\lambda|)$$ 1

Divide by $\lambda > 0$ gives

$$0 \leq \nabla f(\bar{x})^T (x - \bar{x}) + \frac{o(\lambda)}{\lambda}$$

Letting $\lambda \downarrow 0$ yields $\nabla f(\bar{x})^T (x - \bar{x}) \geq 0$ 1



Ex 2 a) P: $\min -x$ st $x-1 \leq 0$

- Obviously sol. of (P) is $\bar{x} = 1$ with $v(P) = -1$
- WD: with $\mathcal{L}(x|y) = -x + y(x-1)$ $y \geq 0$

(WD) $\max\limits_{\substack{y \geq 0 \\ x}} -x + y(x-1)$ 1

st. $\mathcal{L}_x = -1 + y = 0$ or $y = 1$ 1/2

$\max\limits_{x} -x + (x-1) = -1$

indep. of x.

So any $(x, 1)$, $x \in \mathbb{R}$ are feasible and opt. 1

for WD with $\mathcal{L}(x, 1) = -1 = v(WD) = v(P)$ 1

b)

WD: $\max\limits_{y \geq 0, x} \mathcal{L}(x, y) = f(x) + \sum_j y_j g_j(x)$ 1/2

st $\nabla_x \mathcal{L} = \nabla f(x) + \sum y_j \nabla g_j(x) = 0$

$(\bar{x}, \bar{y})$ KKT, i.e., $\bar{x} \in F$, $\bar{y} \geq 0$

$\nabla f(\bar{x}) + \sum \bar{y}_j \nabla g_j(\bar{x}) = 0$ 1/2

$\bar{y}_j \cdot g_j(\bar{x}) = 0$ $\forall j$

So $(\bar{x}, \bar{y})$ is feasible for (WD) with 1

$\mathcal{L}(\bar{x}, \bar{y}) = f(\bar{x}) + \sum_j \underbrace{\bar{y}_j \cdot g_j(\bar{x})}_{=0} = f(\bar{x})$ 1

By weak (WD) duality $(\bar{x}, \bar{y})$ is sol. of (WD) 1

c) Note: $\forall (\bar{x}, \bar{y})$ feas. for (WD) $\bar{y} \geq 0$, $\bar{x}$ is min of $\min\limits_{x \in \mathbb{R}^n} \mathcal{L}(x, \bar{y})$, i.e.,

$\mathcal{L}(\bar{x}, \bar{y}) = \min\limits_{x \in \mathbb{R}^n} \mathcal{L}(x, \bar{y}) \leq \max\limits_{y \geq 0} \min\limits_{x \in \mathbb{R}^n} \mathcal{L}(x, y) = v(D)$

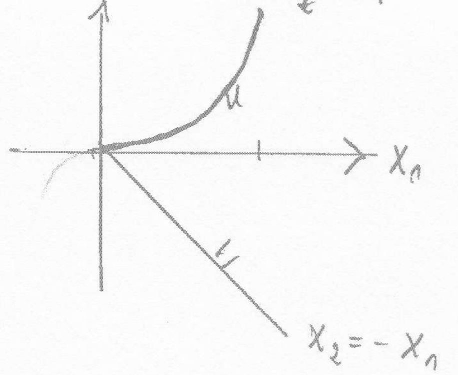
3

So if (WD) is feas. $\Rightarrow$ $v(WD) \leq v(D)$

" " " unfeas $\Rightarrow$ $-\infty = v(WD) \leq v(D)$

$J\bar{x} = \emptyset$: ok

a)



$J\bar{x} = \{1\}$: $\nabla g_1 = \begin{pmatrix} -3x_1^2 \\ 1 \end{pmatrix}$ l.i

$J\bar{x} = \{2\}$: $\nabla g_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ l.i. 1

$J\bar{x} = \{1,2\}$ l.i. $\bar{x} = 0,0$

$\nabla g_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\nabla g_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ l.i 1

7. a) KKT $\nabla f + \mu_1 \nabla g_1 + \mu_2 \nabla g_2 = 0$ $g_1 = g_2 = 0$

$$\begin{pmatrix} 2(x_1+1) \\ 2x_2 \end{pmatrix} + \mu_1 \begin{pmatrix} -3x_1^2 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ 4

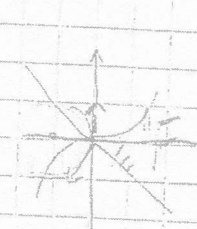
at $\bar{x} = (0,0)$

$2 - \mu_2 = 0 \rightarrow \mu_2 = 2$

$\mu_1 - \mu_2 = 0$ $\mu_1 = 2$

yes!

b) $\nabla g_1(\bar{x}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\nabla g_2(\bar{x}) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ l.i. ind



By Th. 12.2 $\bar{x}$ is loc. min. order 1. $1\frac{1}{2}$

$f(\bar{x}) = 1$

Unique: • see sketch (1)

• $x \in F \Rightarrow \left.\begin{matrix} x_2 \leq x_1^3 \\ -x_2 \leq x_1 \end{matrix}\right\} \Rightarrow \begin{matrix} 0 \leq x_1 + x_1^3 \\ 0 \leq x_1(1+x_1^2) \end{matrix}$ $1\frac{1}{2}$

So $x \in F \Rightarrow f(x) \geq 1 = f(\bar{x})$ $\Rightarrow x_1 \geq 0$ (1)

> 1 $\forall x \in F$ $x \neq (0,0)$ (1/2)

[illegible] $x \in F$, $x+(0,0) \Rightarrow x > 0$

4. Let $\mathcal{K}_1 \subset \mathbb{R}^n$ be a proper cone and let $A \in \mathbb{R}^{n \times n}$ be given. [4 points]

   Show that if A has (full) rank n, then $\mathcal{K}_2 = \{A\mathbf{x} \mid \mathbf{x} \in \mathcal{K}_1\}$ is a proper cone.
   You may assume that $\mathcal{K}_2$ is closed.

Solution: A closed set $\mathcal{K}_2$ is a proper cone if the following hold:

<u>It is a convex cone:</u>
Consider an arbitrary $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{K}_2$ and $\theta, \lambda \geq 0$.
We wish to show that we then have $(\theta\mathbf{y}_1 + \lambda\mathbf{y}_2) \in \mathcal{K}_2$.
$\exists \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{K}_1$ such that $\mathbf{y}_1 = A\mathbf{x}_1$ and $\mathbf{y}_2 = A\mathbf{x}_2$.
As $\mathcal{K}_1$ is a proper cone, we have $(\theta\mathbf{x}_1 + \lambda\mathbf{x}_2) \in \mathcal{K}_1$. 1
Therefore $(\theta\mathbf{y}_1 + \lambda\mathbf{y}_2) = A(\theta\mathbf{x}_1 + \lambda\mathbf{x}_2) \in \mathcal{K}_2$.

<u>It is pointed:</u>
Suppose $\exists \mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{y}, -\mathbf{y} \in \mathcal{K}_2$.
We wish to show that we then have $\mathbf{y} = \mathbf{0}$.
$\exists \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{K}_1$ such that $\mathbf{y} = A\mathbf{x}_1$ and $-\mathbf{y} = A\mathbf{x}_2$.
We have $\mathbf{0} = \mathbf{y} + (-\mathbf{y}) = A(\mathbf{x}_1 + \mathbf{x}_2)$. $1\frac{1}{2}$
As A is rank n, this implies that $\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{0}$.
Therefore we have $\mathbf{x}_1 \in \mathcal{K}_1$ and $-\mathbf{x}_1 = \mathbf{x}_2 \in \mathcal{K}_1$.
As $\mathcal{K}_1$ is a proper cone, this implies that $\mathbf{x}_1 = \mathbf{0}$, and thus $\mathbf{y} = A\mathbf{x}_1 = \mathbf{0}$.

<u>It is full dimensional:</u> Two possible proofs:

1. Suppose for the sake of contradiction that $\mathcal{K}_2$ is not full dimensional.
   Then there exists $\mathbf{z} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\mathbf{z}^T\mathbf{y} = 0$ for all $\mathbf{y} \in \mathcal{K}_2$.
   For all $\mathbf{x} \in \mathcal{K}_1$ we have $A\mathbf{x} \in \mathcal{K}_2$, and thus $0 = \mathbf{z}^T(A\mathbf{x}) = (A^T\mathbf{z})^T\mathbf{x}$.
   As A is rank n we have $A^T\mathbf{z} \neq \mathbf{0}$, and thus we get the contradiction that $\mathcal{K}_1$ is not full-dimensional. $1\frac{1}{2}$
2. As $\mathcal{K}_1$ is full dimensional, $\exists$ linearly independent vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathcal{K}_1$.
   Let $\mathbf{y}_i = A\mathbf{x}_i \in \mathcal{K}_2$ for $i = 1, \ldots, n$.
   As A is rank n and $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent vectors, we have that $\mathbf{y}_1, \ldots, \mathbf{y}_n \in \mathcal{K}_2$ are linearly independent vectors.
   This implies that $\mathcal{K}_2$ is full dimensional.

<u>K_2 closed:</u> $z_k = A x_k \to \bar{z}$, $x_k \in K_1$ Show $\bar{z} = A\bar{x}$, $\bar{x} \in K_1$ Pf: $x_k = A^{-1} z_k \to A^{-1}\bar{z} =: \bar{x} \in K_1$ closed, so $\bar{z} = A\bar{x}$

5. Consider the following one dimensional optimisation problem:

$$\begin{aligned} \min_x \quad & 2x^2 - 2x \\ \text{s.t.} \quad & x^2 \geq 1 \end{aligned} \tag{1}$$

   (a) Sketch this problem. Using this sketch find its optimal solution, x^*, and its optimal value, v(1). [2 points]

   (b) Give the standard sum-of-squares approximation for this problem with $d = 2$. [3 points]

$K_2 = \{y = Ax \mid x \in K_1\}$

$A\ K_1 \to K_2$ $y = Ax \Leftrightarrow x = A^{-1}y$

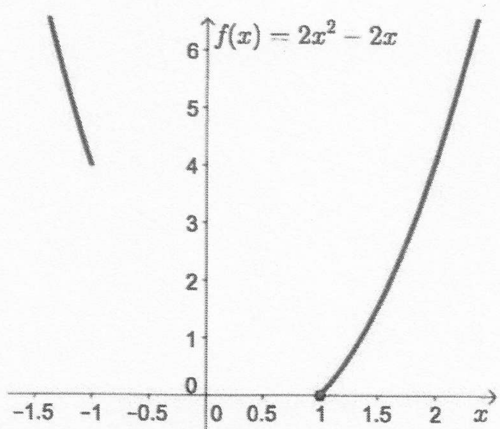
$A^{-1}\ K_2 \to K_1$ K_2 inverse image of ~~conv~~ closed K_1 wrt contin. A^{-1}

(c) For a degree two polynomial $h_0(x) = ax^2 + bx + c$, give a positive semidefinite constraint which is equivalent to the constraint that $h_0 \in \Sigma_2$. [3 points]
This is similar to the fact that for a degree zero polynomial $h_1(x) = a$, we have that $h_1 \in \Sigma_0$ if and only if $a \geq 0$.

(d) Given that $(x-1)^2 \in \Sigma_2$ and $1 \in \Sigma_0$, find a lower bound on the optimal value of the problem from part (b). [1 point]

Solution:

(a) We have that x is feasible if and only if either $x \geq 1$ or $x \leq -1$.
A sketch of the objective function over the feasible set is as follows:



1

From this we see that the optimal solution is at $x^* = 1$.
The optimal value is thus $v(1) = f(x^*) = 0$. 1

(b) In standard form the problem is:

$$\begin{aligned} \min_{x} \quad & 2x^2 - 2x \\ \text{s.t.} \quad & x^2 - 1 \geq 0, \end{aligned}$$

1/2

i.e. $m = n = 1$ and $f(x) = 2x^2 - 2x$ and $g_1(x) = x^2 - 1$.
We have $d_0 = 0$ and $d_1 = 2$ and $d = 2$. The SOS problem is thus:

$$\begin{aligned} \max_{h,t} \quad & t \\ \text{s.t.} \quad & 2x^2 - 2x - t = h_0(x) + h_1(x)\,(x^2 - 1), \\ & h_0 \in \Sigma_2, \quad h_1 \in \Sigma_0. \end{aligned}$$

2 1/2

N.B. 'Σ_i' denotes the set of sum-of-squares polynomials of degree up to and including i. The equality in the SOS optimisation problem is coefficientwise.

(c) We have $n = 1$ and $d = 2$. Therefore $s = \lceil d/2 \rceil = 1$ and $N = \binom{2}{1} = 2$ and $\mathbf{v}_s = \mathbf{v}_1 = (x, \ 1)^T$ (should be of dimension $N = 2$). 1

Considering $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$, we have $\mathbf{v}_s^T A \mathbf{v}_s = a_{11}x^2 + 2a_{12}x + a_{22}$.

Therefore $ax^2 + bx + c = \mathbf{v}_s^T A \mathbf{v}_s$ if and only if $a_{11} = a$ and $a_{12} = \frac{1}{2}b$ and $a_{22} = c$. 1

This implies that $h_0 \in \Sigma_2$ if and only if $\begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix} \in \mathcal{PSD}^2$. 1

(d) Considering $h_0(x) = (x-1)^2$ and $h_1(x) = 1$, we have

$$h_0(x) + h_1(x)\,(x^2 - 1) = (x-1)^2 + (x^2-1) = 2x^2 - 2x - 0.$$ 1

Therefore a feasible point to the problem is $h_0(x) = (x-1)^2$ and $h_1(x) = 1$ and $t = 0$. This implies a lower bound to the problem of 0.

$(x,1)\, A\, (x,1) = a_{11}x^2 + 2a_{12}x + a_{22}$

6. (Automatic additional points) [4 points]

Question:	1	2	3	4	5	6	Total
Points:	4	10	9	4	9	4	40

A copy of the lecture-sheets may be used during the examination.
Good luck!