Retake Exam: Continuous Optimisation 2014

3TU- and LNMB-course, Utrecht.

Thursday 26th February 2015

1. Consider with compact, convex $\mathcal{F} \subset \mathbb{R}^n$ and convex function $f : \mathcal{F} \to \mathbb{R}$ the maximization problem: [3 points]

$$(P) \quad \max f(x) \quad \text{s.t.} \quad x \in \mathcal{F}.$$

Show that the maximum value of (P) is attained (also) at an extreme point of $\mathcal{F}$.

2. Let $f : C \to \mathbb{R}$, be a strictly convex function on the convex set $C \subset \mathbb{R}^n$, *i.e.*, for all $x, y \in C$, $x \neq y$, and $0 < \lambda < 1$ we have [4 points]

$$f((1-\lambda)x + \lambda y) < (1-\lambda)f(x) + \lambda f(y)$$

Show: If $\overline{x} \in C$ is a local minimizer of f then $\overline{x}$ is the unique global minimizer of f on C.

3. *[Relation KKT-condition and saddle point condition]* Consider the convex program

$$(CO) \qquad \min f(x) \quad \text{s.t.} \quad x \in \mathcal{F} := \{x \in \mathbb{R}^n \mid g_j(x) \leq 0,\ j = 1, \ldots, m\},$$

with convex functions $f, g_j \in C^1(\mathbb{R}^n, \mathbb{R})$ (and $C = \mathbb{R}^n$).

   (a) Give the Lagrangian function $L(x, y)$ of (CO). [1 point]

   (b) Show: If $(\overline{x}, \overline{y})$ is a saddle point of $L(x, y)$ then $\overline{x}$ satisfies the KKT-conditions for (CO) with Lagrange multiplier vector $\overline{y}$. [4 points]

   (c) Show: If $\overline{x}$ satisfies the KKT-conditions for (CO) with Lagrange multiplier vector $\overline{y}$ then $(\overline{x}, \overline{y})$ is a saddle point of $L(x, y)$. [4 points]

4. Consider the constrained minimization problem:

$$(P): \qquad \min_{x \in \mathbb{R}^2} (-2x_1 - 4x_2) \quad \text{s.t.} \quad x_1^2 + 2x_2^2 \leq 1\,.$$

   (a) Compute a (the) solution(s) $\overline{x}$ of the system of KKT-conditions of (P). [3 points]

   (b) Give a geometrical sketch of the problem (indicate the feasible set, objective vector, KKT-condition at $\overline{x}$). [2 points]

   (c) Is the solution $\overline{x}$ in (a) a (global) minimizer of (P)? Is $\overline{x}$ a minimizer of order 1 or of order 2? (Explain in detail!) [3 points]

5. In this question, we consider a hyperplane $\mathcal{H} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^\mathsf{T}\mathbf{x} = b\}$ with fixed $\mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $b \in \mathbb{R} \setminus \{0\}$. We wish to find the distance between the origin and the closest point in this hyperplane.
   (a) Write this problem as an optimisation problem in standard form over the second order cone. [2 points]
   (b) Give the dual problem to this in standard form. [2 points]
   (c) Simplify the dual problem and give the optimal value in terms of $\|\mathbf{a}\|_2$ and b. [2 points]

6. We will consider bounds to the optimal value of the following problem:

$$\begin{aligned} \min_{x} \quad & x^4 + 2x^3 - 3x^2 + 1 \\ \text{s.t.} \quad & x \in \mathbb{R}. \end{aligned} \tag{A}$$

   (a) Give an upper bound on the optimal value of problem (A). [1 point]
   (b) Formulate a sum-of-squares optimisation problem to give a lower bound on the optimal value of problem (A). [2 points]
   (c) For fixed $u \in \mathbb{R}$, consider the polynomial $f(x) = x^4 + 2x^3 - 3x^2 + u$. Write the constraint that f is a sum-of-squares polynomial explicitly as a positive semidefinite constraint. [3 points]
   Hint: You will need to introduce new variable(s) in order to do this.

7. (Automatic additional points) [4 points]

Question:	1	2	3	4	5	6	7	Total
Points:	3	4	9	8	6	6	4	40

A copy of the lecture-sheets may be used during the examination.
Good luck!