Exam: Continuous Optimisation 2014

3TU- and LNMB-course, Utrecht.

Monday 26th January 2015

1. Given a convex set $\mathcal{F} \subset \mathbb{R}^n$ and a convex C^1-function $f : \mathcal{F} \to \mathbb{R}$, consider the program: [4 points]

$$(P) \quad \min f(x) \quad \text{s.t.} \quad x \in \mathcal{F}.$$

Show for $\overline{x} \in \mathcal{F}$:

$\overline{x}$ is a (global) minimizer of (P) if and only if $\nabla f(\overline{x})^T(x - \overline{x}) \geq 0 \ \forall x \in \mathcal{F}$ holds.

2. (a) Consider the simple linear program: [3 points]

$$(P) \quad \min_{x \in \mathbb{R}} -x \quad \text{s.t.} \quad x - 1 \leq 0\,.$$

Look at the Wolfe dual (WD) of (P) and determine all solutions $(\overline{x}, \overline{y})$ of (WD). Prove in this way that strong duality, $v(WD) = v(P)$, holds and show that not all solutions $(\overline{x}, \overline{y})$ of (WD) correspond to KKT points of (P) (not all points $\overline{x}$ are feasible).

(b) For the convex program [4 points]

$$(CO) \qquad \min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad g_j(x) \leq 0,\ j = 1, \ldots, m\,,$$

with convex functions $f, g_j \in C^1(\mathbb{R}^n, \mathbb{R})$ show:

If the feasible point $\overline{x}$ satisfies the KKT-conditions with a multiplier vector $\overline{y} \geq 0$ then $(\overline{x}, \overline{y})$ is a solution of the Wolfe dual (WD).

(c) For the program (CO) in (b) show for the values $v(WD)$ of Wolfe's dual and $v(D)$ of the Lagrangean dual that we have: $\quad v(WD) \leq v(D)$ [3 points]

3. Consider the problem:

$$(P) \quad \min_{x \in \mathbb{R}^2} (x_1 + 1)^2 + x_2^2 \quad \text{s.t.} \quad \begin{array}{r} x_2 - x_1^3 \leq 0 \\ -x_1 - x_2 \leq 0 \end{array}$$

(a) Sketch the feasible set $\mathcal{F}$ of (P). Show that at any feasible point $x \in \mathcal{F}$ the linear independency constraint qualification (LICQ) holds. [2 points]

(b) Show that for $\overline{x} = (0, 0)$ the Karush-Kuhn-Tucker conditions are satisfied. [4 points]

(c) Show that $\overline{x} = (0, 0)$ is a strict local minimizer of order $p = 1$. Also prove (in detail) that $\overline{x}$ is the unique global minimizer? [3 points]

4. Let $\mathcal{K}_1 \subset \mathbb{R}^n$ be a proper cone and let $A \in \mathbb{R}^{n\times n}$ be given. [4 points]

   Show that if A has (full) rank n, then $\mathcal{K}_2 = \{A\mathbf{x} \mid \mathbf{x} \in \mathcal{K}_1\}$ is a proper cone.
   You may assume that $\mathcal{K}_2$ is closed.

5. Consider the following one dimensional optimisation problem:

$$\begin{aligned} \min_x \quad & 2x^2 - 2x \\ \text{s.t.} \quad & x^2 \geq 1 \end{aligned} \tag{1}$$

   (a) Sketch this problem. Using this sketch find its optimal solution, x^*, and its optimal value, v(1). [2 points]

   (b) Give the standard sum-of-squares approximation for this problem with $d = 2$. [3 points]

   (c) For a degree two polynomial $h_0(x) = ax^2 + bx + c$, give a positive semidefinite constraint which is equivalent to the constraint that $h_0 \in \Sigma_2$. [3 points]
   This is similar to the fact that for a degree zero polynomial $h_1(x) = a$, we have that $h_1 \in \Sigma_0$ if and only if $a \geq 0$.

   (d) Given that $(x-1)^2 \in \Sigma_2$ and $1 \in \Sigma_0$, find a lower bound on the optimal value of the problem from part (b). [1 point]

6. (Automatic additional points) [4 points]

Question:	1	2	3	4	5	6	Total
Points:	4	10	9	4	9	4	40

A copy of the lecture-sheets may be used during the examination.
Good luck!