Optimal Control (course code: 191561620)

Date:	21-01-2026
Place:	SP 5
Time:	08:45–11:45 (till 12:30 for students with special rights)
Course coordinator:	G. Meinsma
Allowed aids during test:	NONE

0. Which MSc programme (AM, BME, EE, SC, ?) do you do?

1. Determine all points of equilibrium of the system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \cos(x_1(t)(1-x_2(t))) \\ x_2(t)+\sin(2x_1(t)) \end{bmatrix}.$$

2. Determine a Lyapunov function for

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

at equilibrium $\bar{x} = (0,0)$.

3. Consider the cost and boundary conditions

$$\int_0^1 x^2(t) + \dot{x}^2(t) + 4tx(t)\,dt, \qquad x(0) = 0, x(1) = 2.$$

   (a) Determine the function $x_*(t)$ that satisfies the Euler-Lagrange equation for this cost function and that satisfies the the given boundary conditions.

   (b) Is the solution found in (a) a global optimal solution?

4. Consider the system

$$\dot{x}(t) = x(t) + u(t),$$

with initial condition $x(0) = x_0$, and cost

$$\int_0^1 \tfrac{1}{2}u^2(t) - 2u(t) - 2x(t)\,dt,$$

and let $\mathbb{U} = [0,4]$ so the input is restricted.

   (a) Determine the Hamiltonian.

   (b) Determine the costate differential equation, including final condition.

   (c) Determine, using the minimum principle, the optimal control $u_* : [0,1] \to [0,4]$ as a function of time.

5. Consider the optimal control problem

$$\dot{x}(t) = x(t)(1-u(t)), \qquad x(0) = x_0, \qquad \mathbb{U} = [0,\infty)$$

and cost

$$J_{[0,T]}(x_0,u) = -\sqrt{x(T)} + \int_0^T -\sqrt{x(t)u(t)}\,dt.$$

Throughout we assume that $x_0 > 0$. It can be shown that then $x(t) \geq 0$ for every bounded u.

(a) Assume that the HJB equation has a solution of the form $V(x,t) = -\sqrt{Q(t)x}$ for some function $Q(t)$. Derive an ordinary differential equation for $Q(t)$ including final condition. The differential equation must not depend on x and u.

(b) It can be shown that $Q(t) = 2e^{T-t} - 1$. Knowing that, determine an optimal control $u_*(t)$ in terms of $Q(t), x(t)$, and argue that u_* is optimal (so not just a candidate optimal control).

(c) Determine the optimal cost $J_{[0,T]}x_0, u_*)$.

6. Consider the LQ problem-with-stability, with

$$\dot{x}(t) = ax(t) + u(t), \quad x(0) = x_0 \in \mathbb{R}, \quad \mathbb{U} = \mathbb{R}, \quad J_{[0,\infty)}(x_0,u) = \int_0^\infty x^2(t) + u^2(t)\,dt$$

Here, a is an arbitrary real number.

(a) Determine F such that $u(t) = -Fx(t)$ solves the infinite-horizon LQ problem with stability.

(b) Let P be the "appropriate" solution of the ARE. Explain in words why you are not surprised that $\lim_{a\to-\infty} x_0^2 P = 0$.

problem:	1	2	3	4	5	6
points:	3	4	4+2	1+2+5	5+3+2	3+2

Exam grade: $1 + 9\frac{p}{p_{\max}}$.

Euler-Lagrange eqn: $\left(\frac{\partial}{\partial x} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\right) F(t, x(t), \dot{x}(t)) = 0$

Beltrami identity: $F(x,\dot{x}) - \dot{x}^T \frac{\partial F(x,\dot{x})}{\partial \dot{x}} = C$

Standard Hamiltonian eqn: $\dot{x} = \frac{\partial H(x,p,u)}{\partial p}$, $x(0) = x_0$ & $\dot{p} = -\frac{\partial H(x,p,u)}{\partial x}$, $p(T) = \frac{\partial K(x(T))}{\partial x}$

HJB eqn: $\frac{\partial V(x,t)}{\partial t} + \min_{u\in\mathbb{U}}\left(\frac{\partial V(x,t)}{\partial x^T} f(x,u) + L(x,u)\right) = 0, \quad V(x,T) = K(x)$

LQ Riccati differential eqn: $\dot{P}(t) = -P(t)A - A^T P(t) + P(t)BR^{-1}B^T P(t) - Q, \quad P(T) = S$