Examination: Mathematical Programming I (158025)

July 1, 2002, 13.30-16.30

Ex.1 Consider the primal-dual pair of linear problems:

$$\begin{array}{lll} (P) & \max\limits_{\mathbf{x}\in\mathbb{R}^n} \mathbf{c}^T\mathbf{x} \quad \text{s.t.} & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ (D) & \min\limits_{\mathbf{y}\in\mathbb{R}^m} \mathbf{b}^T\mathbf{y} \quad \text{s.t.} & \mathbf{A}^T\mathbf{y} = \mathbf{c}\ , \quad \mathbf{y} \ge \mathbf{0} \end{array}$$

Assume (P) and (D) are both feasible.

(a) Let $\mathbf{x}$ be a (fixed) vector, feasible for (P). Show that there exists a constant $\kappa \ge 0$ such that for any optimal solution $\mathbf{y}$ of (D) we have:

$$0 \le \mathbf{b}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} = \mathbf{y}^T(\mathbf{b} - \mathbf{A}\mathbf{x}) = \kappa\ .$$

(b) Show that the set of optimal solutions of (D) is bounded if and only if (P) has a strictly feasible point $\mathbf{x}$ (*i.e.*, $\mathbf{A}\mathbf{x} < \mathbf{b}$).

Hint: " $\Leftarrow$": Use (a)

Ex. 2

(a) Show that $f(x) = -\ln x$ is a convex function on $(0, \infty)$.

(b) Use (a) to show (*geometric and arithmetic mean*):

$$(a_1 a_2 \cdots a_n)^{1/n} \le \frac{a_1 + a_2 + \ldots + a_n}{n} \qquad \text{for all } a_1, \ldots, a_n > 0\ .$$

Ex. 3 Let $\mathbf{A}$ be a positive definite matrix and $\mathbf{b} \in \mathbb{R}^n$.

(a) Show that $\langle \mathbf{x} | \mathbf{y} \rangle_A := \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{y}$ defines an inner product on $\mathbb{R}^n$.
So $\|\mathbf{x}\|_A := \sqrt{\langle \mathbf{x} | \mathbf{x} \rangle_A}$ defines a norm on $\mathbb{R}^n$.

(b) Let $C \subseteq \mathbb{R}^n$ be a closed convex set. Consider the minimization problem.

$$(P) \qquad \min \left\{ \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} \ \mid\ \mathbf{x} \in C \right\}$$

Show: A minimizer of (P) exists and is unique.

Ex. 4 Find the critical points (i.e. points satisfying $\nabla f(\mathbf{x}) = \mathbf{0}^T$, $\mathbf{x} = (x_1, x_2)$) of the function

$$f(\mathbf{x}) = -(x_1 - x_2)^4 + x_1^2 + x_2^2$$

and determine the local minimizers.
Does there exist a global minimizer or a global maximizer of f on $\mathbb{R}^n$?

Ex. 5 For the function $f : \mathbb{R}^n \to \mathbb{R}$, $f(\mathbf{x}) = \|\mathbf{x}\|$ (Euclidean norm) show:

(a) f is convex and continuous on $\mathbb{R}^n$.

(b) For any $\overline{\boldsymbol{x}} \in \mathbb{R}^n$

$$\partial f(\overline{\boldsymbol{x}}) = \begin{cases} \{\boldsymbol{\xi} \in \mathbb{R}^n \mid \|\boldsymbol{\xi}\| \le 1\} & \text{if } \overline{\boldsymbol{x}} = \mathbf{0} \\ \{\overline{\boldsymbol{x}}/\|\overline{\boldsymbol{x}}\|\} & \text{if } \overline{\boldsymbol{x}} \neq \mathbf{0} \end{cases}$$

Ex. 6 Let be given convex functions $f_j \in C^1(\mathbb{R}^n, \mathbb{R})$, $j \in J$, with a finite index set J.
Define the function $f(\mathbf{x}) := \max\{f_j(\mathbf{x}) \mid j \in J\}$, $\mathbf{x} \in \mathbb{R}^n$ and for any $\overline{\boldsymbol{x}} \in \mathbb{R}^n$ the set $J(\overline{\boldsymbol{x}}) = \{j \in J \mid f(\overline{\boldsymbol{x}}) = f_j(\overline{\boldsymbol{x}})\}$.

(a) Show that the function f is convex on $\mathbb{R}^n$.

(b) Prove that for any $\overline{\boldsymbol{x}} \in \mathbb{R}^n$ we have :

$$\operatorname{conv}\{\nabla f_j(\overline{\boldsymbol{x}}) \mid j \in J(\overline{\boldsymbol{x}})\} \quad \subset \quad \partial f(\overline{\boldsymbol{x}})\ .$$

Points: 36+4 =40

Ex. 1	a	:	2 pt.
	b	:	5 pt.
Ex. 2	a	:	2 pt.
	b	:	3 pt.
Ex. 3	a	:	2 pt.
	b	:	4 pt.
Ex. 4		:	6 pt.
Ex. 5	a	:	2 pt.
		b :	4 pt.
Ex. 6	a	:	3 pt.
	b	:	3 pt.

The scipt ’Mathematical Programming I’
may be used during the examination. Good luck!