Exam Mathematical Optimisation (201500379)
Tuesday 16th April, 8.45 – 11.45

Motivate all your answers!

A copy of the lecture-sheets may be used during the exam. Good Luck!

1. Let $A \in \mathbb{R}^{n \times n}$ be symmetric. Show that
   (a) $\min\{x^T Ax \mid x^T x = 1\} = \max\{\lambda \mid A - \lambda I \succeq 0\}$, [4 points]
   (b) the use of "min" and "max" in (a) is justified, [2 points]
   (c) the optimal value in (a) is the smallest eigenvalue of A. [2 points]
   (Hint: You may use any result (Theorem, Lemma) from the sheets.)

2. Compute a lattice basis (for the colomns) of $A = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 1 & 2 \end{bmatrix}$. [4 points]

3. (a) Prove from scratch (i.e., without referring to any results from the sheets) that weak duality holds in Linear Programming. [3 points]
   (b) Interpret the following "Theorem of the Alternative" geometrically: [2 points]
   Given $A \in R^{m \times n}, b \in \mathbb{R}^m$, exactly one of the following is true:
   - $\exists x \in \mathbb{R}^n : Ax = b, x \geq 0$
   - $\exists y \in \mathbb{R}^m : y^T A \geq 0, y^T b < 0$.

4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1–function.
   (a) Show that f is convex if and only if for all $x \in \mathbb{R}^n$ and all $h \in \mathbb{R}^n$ the corresponding C^1–function $p(\lambda) := f(x + \lambda h)$ is a convex function on $[0, 1]$. [2 points]
   (b) Show that f is convex if and only if for all $x \in \mathbb{R}^n$ and all $h \in \mathbb{R}^n$ [5 points]
   $$[\nabla f(x + h) - \nabla f(x)]^T h \geq 0.$$
   (Hint: One may use Theorem 4.8 from the sheets for "$\Rightarrow$" and apply Theorem 4.7 to the one variable functions $p(\lambda)$ to prove "$\Leftarrow$".)

5. Consider $f(x_1, x_2) = x_1^4 + 2x_1 x_2 + 2(x_1 + x_2) + x_2^2$.
   (a) Determine the critical points and local minimizers of f. [4 points]
   (b) Does f have global minima? [2 points]

6. For the Quasi-Newton Method, show that
   (a) $d_k = -H_k g_k$ is a descent direction, provided $H_k \succ 0$. [3 points]
   (b) Explain why at the next iteration point x_{k+1} we have $g_{k+1}^T d_k = 0$. [3 points]