Exam: Mathematical Optimisation

08:45 – 11:45, Tuesday 17th April 2018

Hints at the end of the paper. Workings must be shown. Good Luck!

1. Let $\mathsf{A} \in \mathcal{S}^n$ be positive semidefinite. For $\mathbf{x} \in \mathbb{R}^n$ show that the following implication holds: [3 points]

$$\mathbf{x}^\mathsf{T}\mathsf{A}\mathbf{x} = 0 \qquad \Leftrightarrow \qquad \mathsf{A}\mathbf{x} = \mathbf{0}.$$

Hint: Use Corollary 2.7 from the slides.

2. For the following system of inequalities, either find a feasible solution or show that a feasible solution does not exist. [3 points]

$$-5x_1 + x_2 \leq 3, \qquad \text{(A)}$$
$$-5x_1 - 2x_2 - 3x_3 \leq -2, \qquad \text{(B)}$$
$$5x_1 + x_3 \leq -2 \qquad \text{(C)}$$

3. Consider the pair of primal-dual linear programs:

$$\max_{\mathbf{x}\in\mathbb{R}^n} \quad \mathbf{c}^\mathsf{T}\mathbf{x} \quad \text{s.t.} \quad \mathsf{A}\mathbf{x} \leq \mathbf{b}, \qquad \text{(D)}$$
$$\min_{\mathbf{y}\in\mathbb{R}^m} \quad \mathbf{b}^\mathsf{T}\mathbf{y} \quad \text{s.t.} \quad \mathsf{A}^\mathsf{T}\mathbf{y} = \mathbf{c},\ \mathbf{y} \geq 0. \qquad \text{(E)}$$

(a) Show that weak duality holds for these problems, i.e. for any pair $\mathbf{x}, \mathbf{y}$ with $\mathbf{x}$ feasible for (D) and $\mathbf{y}$ feasible for (E), it holds that [2 points]

$$\mathbf{c}^\mathsf{T}\mathbf{x} \leq \mathbf{b}^\mathsf{T}\mathbf{y}.$$

From now on in this question let (E) have a feasible point, and let ν be the minimal value of problem (E).

(b) Using part (a) of this question, show that if $\nu = -\infty$ then problem (D) does not have a feasible point. [1 point]

(c) Using Farkas' lemma (Theorem 2.12 from the slides, Theorem 2.6 from the reader), show that if problem (D) does not have a feasible point then $\nu = -\infty$. [3 points]

4. For a finite set $\mathcal{J}$, let $f_i \in C^1(\mathbb{R}^n, \mathbb{R})$ be a convex function for all $i \in \mathcal{J}$, and define the function $f(\mathbf{x}) := \max\{f_i(\mathbf{x}) : i \in \mathcal{J}\}$.

(a) Show that the function f is convex on $\mathbb{R}^n$. [3 points]

(b) For $\overline{\mathbf{x}} \in \mathbb{R}^n$, define the set $\mathcal{J}(\overline{\mathbf{x}}) = \{j \in \mathcal{J} : f(\overline{\mathbf{x}}) = f_j(\overline{\mathbf{x}})\}$. Show that for all $\overline{\mathbf{x}} \in \mathbb{R}^n$ we have [3 points]

$$\operatorname{conv}\{\nabla f_i(\overline{\mathbf{x}}) : i \in \mathcal{J}(\overline{\mathbf{x}})\} \subseteq \partial f(\overline{\mathbf{x}}).$$

(c) Show that the point $\overline{\mathbf{x}} = \begin{pmatrix}0, & 1\end{pmatrix}^\mathsf{T}$ is a global minimiser of $f(\mathbf{x})$ over $\mathbb{R}^n$, where [3 points]

$$f(\mathbf{x}) = \max\{\exp(x_1) + x_2^2,\ 6 - 2x_1 - 4x_2,\ \exp(-x_1) - 2x_2\}.$$

5. (a) Show that $f(x) = -\ln(x)$ is a convex function on $(0, \infty)$. [1 point]
   (b) Show that for any set of values $p_1, \ldots, p_n > 0$ with $\sum_{i=1}^n p_i = 1$ we have [3 points]

$$-\sum_{i=1}^{n} p_i \ln(p_i) \leq \ln(n)$$

6. Consider $f \in C^1(\mathbb{R}^n, \mathbb{R})$, $\mathsf{H} \in \mathcal{S}^n$, $\mathbf{x}_k \in \mathbb{R}^n$, $\mathbf{g}_k = \nabla f(\mathbf{x}_k) \neq \mathbf{0}$ and $\mathbf{d}_k = -\mathsf{H}\mathbf{g}_k$.
   (a) Show that if H is positive definite then $\mathbf{d}_k$ is a descent direction for f from $\mathbf{x}_k$. [1 point]
   (b) Assuming that H is indeed positive definite, let $t_k = \arg\min_{t \geq 0} f(\mathbf{x}_k + t\mathbf{d}_k)$, let $\mathbf{x}_{k+1} = \mathbf{x}_k + t_k\mathbf{d}_k$ and let $\mathbf{g}_{k+1} = \nabla f(\mathbf{x}_{k+1})$. Show that $\mathbf{g}_k^{\mathsf{T}}\mathsf{H}\mathbf{g}_{k+1} = 0$. [2 points]

7. Consider the problem of minimising $f : \mathbb{R}^n \to \mathbb{R}$ over $\mathbb{R}^n$ for

$$f(\mathbf{x}) = 3x_1^4 - 4x_1^3 + 4x_1^2 + 2x_2^2 - 4x_1^2x_2.$$

   (a) What is the gradient vector, $\nabla f(\mathbf{x})$, and the Hessian matrix, $\nabla^2 f(\mathbf{x})$, for this function? [1 point]
   (b) Determine the critical points and local minimiser(s) of f. [2 points]

From now on in this question, for $\mathbf{x}_k, \mathbf{d}_k \in \mathbb{R}^n$, we let $t_k = \arg\min_t\{f(\mathbf{x}_k + t\mathbf{d}_k)\}$ and $\mathbf{x}_{k+1} = \mathbf{x}_k + t_k\mathbf{d}_k$.

Consider $\mathbf{x}_4 = \begin{pmatrix}1, & 0\end{pmatrix}^{\mathsf{T}}$ and $\mathbf{g}_3 = \begin{pmatrix}0, & -4\end{pmatrix}^{\mathsf{T}}$ and $\mathbf{d}_3 = \begin{pmatrix}1, & 2\end{pmatrix}^{\mathsf{T}}$. We will compute the following directions at $\mathbf{x}_4$, which should not be normalised:
   (c) Determine the direction of steepest descent of f at $\mathbf{x}_4$. [1 point]
   (d) Using the Polak-Ribiere formula for α_4, determine the conjugate gradient direction at $\mathbf{x}_4$ [2 points]
   (e) Determine the Newton direction of f at $\mathbf{x}_4$. [2 points]

8. (Automatic additional points) [4 points]

Question:	1	2	3	4	5	6	7	8	Total
Points:	3	3	6	9	4	3	8	4	40

A copy of the lecture-sheets may be used during the examination.

Hints: 1. $\begin{pmatrix} a & b \\ b & c \end{pmatrix}^{-1} = \dfrac{1}{ac - b^2}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$;

2. $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ *is positive semidefinite if and only if* $a + c \geq 0$ *and* $ac \geq b^2$.

3. $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ *is positive definite if and only if* $a + c > 0$ *and* $ac > b^2$.

4. $-\ln(p_i) = \ln(1/p_i)$