Examination: Mathematical Programming I (191580250)

July 1, 2011, 8.45 -11.45

Ex.1 Prove the following statements.

(a) Let $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a positive definite matrix. Then $\mathbf{A}$ is regular and also the inverse $\mathbf{A}^{-1}$ is positive definite.

(b) For a matrix $\mathbf{A} \in \mathbb{R}^{n\times m}$ show: The matrix $\mathbf{B} := \mathbf{A}\cdot\mathbf{A}^T$ is positive semidefinite. Moreover, $\mathbf{B}$ is positive definite if and only if the matrix $\mathbf{A}$ has rank n (*i.e.*, the rows of $\mathbf{A}$ are linearly independent).

Ex.2

(a) Show that the following system does <u>*not*</u> have any feasible solution.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & \leq & -1 \\ -2x_1 & + & x_2 & & & \leq & 2 \\ & & -5x_2 & - & 6x_3 & \leq & -1 \end{array} .$$

(b) Consider the pair of primal and dual linear programs,

$$\begin{array}{llll} \text{P}: & \max\limits_{\mathbf{x}\in\mathbb{R}^n} \mathbf{c}^T\mathbf{x} & \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ \text{D}: & \min\limits_{\mathbf{y}\in\mathbb{R}^m} \mathbf{b}^T\mathbf{y} & \text{s.t.} & \mathbf{A}^T\mathbf{y} = \mathbf{c}\ , \quad \mathbf{y} \geq \mathbf{0}\ , \end{array}$$

where $\mathbf{A}$ is an $(m\times n)$-matrix $(m \geq n)$ and $\mathbf{c} \in \mathbb{R}^n,\ \mathbf{b} \in \mathbb{R}^m$. Let v_P denote the maximum value of the primal program P and v_D the minimum value of the dual problem D. The feasible sets of P and D are abbreviated by F_P and F_D. Suppose the feasible set F_P is not empty ($F_P \neq \emptyset$). Show that then we have

$$F_D = \emptyset \quad \text{if and only if} \quad v_P = \infty\ .$$

Ex.3 Let $S^{n\times n} := \{\mathbf{A} \in \mathbb{R}^{n\times n} \mid \mathbf{A} \text{ is symmetric}\}$.

(a) Show that for fixed $\overline{x}$ the function $g(A) := \overline{x}^T\mathbf{A}\overline{x}$ is a convex function (on $S^{n\times n}$).
(*Note: g is even linear in* $\mathbf{A}$)

(b) Consider now the function $f : S^{n\times n} \to \mathbb{R}$, defined by:

$$f(\mathbf{A}) = \max_{\mathbf{x}\in\mathbb{R}^n, \mathbf{x}^T\mathbf{x}=1} \mathbf{x}^T\mathbf{A}\mathbf{x}\ .$$

Show that $f(\mathbf{A})$ is a convex function (on $S^{n\times n}$).
(Is the function f well-defined?, *i.e.*, is for given $\mathbf{A}$ the maximum value attained?)

Ex. 4 Let $f : \mathbb{R}^n \to \mathbb{R}$, be a C^1-function on $\mathbb{R}^n$. Show that f is convex if and only if the following inequality holds:

$$(\nabla f(x) - \nabla f(x'))\,(x - x') \geq 0 \quad \text{for all } x, x' \in \mathbb{R}^n \;.$$

Hint: For "$\Leftarrow$" use the mean-value relation:
$f(x) - f(x') = \nabla f(x' + \lambda(x - x'))(x - x')$ *for some* $\lambda \in (0, 1)$.

Ex.5 Given the function $f(\mathbf{x}) = x_1^4 + x_2^4 - 4x_1x_2 + 2$.

(a) Determine the critical points and the local minimizer(s) of f.

(b) Does there exist a global minimizer (on $\mathbb{R}^n$).

Ex. 6 We wish to find the minimizer of the quadratic function $q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x}$ with positive definite matrix $\mathbf{A}$.

(a) Show that for any starting point $\mathbf{x}_0$ the Newton method finds the minimizer of q in one step.

(b) Let us apply the Quasi-Newton method. Suppose that this method produces the iterates x_k, the search directions $\mathbf{d}_k$ and the matrices $\mathbf{H}_k,\ k = 0, 1, ..$. Show that the relation holds:

$$\mathbf{H}_k^{-1} d_j = \mathbf{A} d_j \text{ , for all } j = 0, \ldots, k-1 \;,$$

and after n steps we have $\mathbf{H}_n = A^{-1}$.
Hint: Use the relation $\mathbf{H}_k\boldsymbol{\gamma}_j = \boldsymbol{\delta}_j,\ 0 \leq j \leq k-1$ *where* $\boldsymbol{\gamma}_j = \mathbf{g}_{j+1} - \mathbf{g}_j,\ \boldsymbol{\delta}_j = \mathbf{x}_{j+1} - \mathbf{x}_j$.

Points: 36+4 =40

Ex. 1	a :	3 pt.		Ex. 4	a :	6 pt.
	b :	3 pt.		Ex. 5	a :	4 pt.
Ex. 2	a :	3 pt.			b :	2 pt.
	b :	3 pt.		Ex. 6	a :	2 pt.
Ex. 3	a :	2 pt.			b :	4 pt.
	b :	4 pt.				

The scipt 'Mathematical Programming I' and a copy of the lecturesheets may be used during the examination. Good luck!