Exam Complex Function Theory
Code 2023-201500405

Date	:	Wednesday, June 12, 2024
Place	:	NH-205
Time	:	08.45 – 10.45

All answers must be motivated.
The use of the book, or lecture notes, summaries, etc. is not allowed.
The use of a pocket calculator or any other electronic equipment is not allowed.

1. Of each of the following statements determine if it is true or not. If it holds, provide a proof, otherwise show that it does not hold, for instance via a counterexample.

   (a) If $u(x, y)$ is harmonic for all $x, y \in \mathbb{R}$, then $f(z)$ defined as $f(x + iy) = u(x; y) + iu(x; y)$ is entire.

   (b) Let f have the Maclaurin/Taylor series consisting only of even powers i.e., $f(z) = \sum_{k=0}^{\infty} a_k z^2$ for $z \in \mathbb{C}$ satisfying $|z| < 1$, then $g(z) := f(\sqrt{z})$ defines an analytic on the same disc.

   (c) The equation $\sin(z) = 5$ has one solution in $\mathbb{C}$.

   (d) Let f be an analytic function in the punctured disc $\{z \in \mathbb{C} \mid |z| < 1 \text{ and } z \neq 0\}$. Let C_r be the circle with radius r and centre zero. If for all $r \in (0, 1)$ the integral $\int_{C_r} f(z)dz$ is zero, then f is analytic in the disc $\{z \in \mathbb{C} \mid |z| < 1\}$.

2. Consider the "broken" unit circle Γ_ε, which for $\varepsilon \in (0, 1)$ is defined as $\Gamma_\varepsilon = \{e^{i\theta} \mid \theta \in [-\pi + \varepsilon, \pi - \varepsilon]\}$.

   (a) Determine the integral $I(\varepsilon) = \int_{\Gamma_\varepsilon} \text{Log}(z)dz$. Here Γ_ε is traversed in the counter-clockwise direction.
   Hint: Determine the derivative of $z\text{Log}(z) + \beta z$

   (b) Calculate $\lim_{\varepsilon \downarrow 0} I(\varepsilon)$ and explain why it cannot be zero.

P.T.O.

3. Determine the integral of $\frac{1}{z^2-1}$ along the closed curve γ, as drawn in Figure 1, where γ is traversed once in the positive direction.

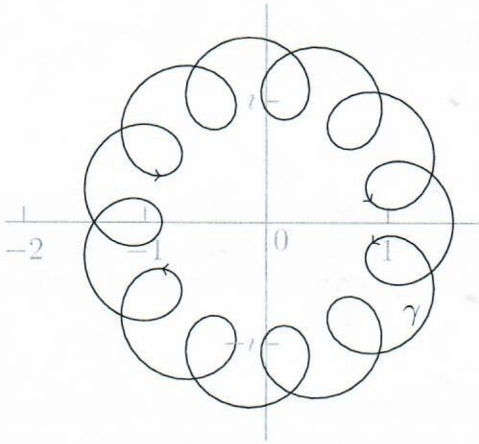

Figure 1: The curve γ.

4. Consider the function

$$h(z) = \frac{z(z^2-1)}{\cos(z)-1}.$$

   (a) Determine the poles of h with their order.
   (b) Determine the zeros of h with their order.
   (c) Is the Laurant series around $z = 0$ of the form $h(z) = \sum_{k=-m}^{M} a_k z^k$, $m, M \in \mathbb{N}$ or of the form $h(z) = \sum_{k=-m}^{\infty} a_k z^k$, $m \in \mathbb{N}$?

5. Determine the integral

$$\int_{-\infty}^{\infty} \frac{x^2+x+1}{(x^2+4)(x^2+2x+10)}\,dx.$$

6. Let f be an entire function, satisfying

$$|f(z)| < |z|, \text{ when } |z| \geq 2.$$

   (a) Show that $f(z) + z$ has exactly one zero (counted with multiplicity) within the circle $C_2 := \{z \in \mathbb{C} \mid |z| = 2\}$
   (b) Show that f is a linear function, i.e., $f(z) = az + b$, for some $a, b \in \mathbb{C}$.

Points[1]

Ex. 1		Ex. 2		Ex. 3	Ex. 4		Ex. 5	Ex. 6	
a	2	a	3	4	a	3	5	a	3
b	2	b	2		b	3		b	3
c	2				c	2			
d	2								

[1]Total: $36 + 4 = 40$ points