# Introduction to Investment Science

## Solution of Final Examination

## 2010-2011

1. (a) Coupons are paid annually. There are still two periods to maturity.

Cash flow stream for X : $(-105, 13, 113)$

Cash flow stream for Y : $(-91.25, 5, 105)$.

(b)
$$PV_X = 13\, d_{0,1} + 113\, d_{0,2}$$
$$PV_Y = 5\, d_{0,1} + 105\, d_{0,2}$$

(c) Equating present value with current price,
$$105 = 13\, d_{0,1} + 113\, d_{0,2}$$
$$91.25 = 5\, d_{0,1} + 105\, d_{0,2}$$

Solving,
$$d_{0,1} = 0.8922$$
$$d_{0,2} = 0.8266.$$

(d)
$$\frac{1}{1+s_1} = d_{0,1} \Rightarrow s_1 = d_{0,1}^{-1} - 1 \cong 12\%$$
$$\frac{1}{(1+s_2)^2} = d_{0,2} \Rightarrow s_2 = d_{0,2}^{-2} - 1 \cong 10\%$$

(e)
$$(1+s_1)(1+f_{1,2}) = (1+s_2)^2$$
$$\Rightarrow \quad 1 + f_{1,2} = \frac{(1+s_2)^2}{1+s_1} = \frac{d_{0,1}}{d_{0,2}} = 1.0794.$$

If I invest € 1000 a year from now, I expect to get $1000 \times (1+f_{1,2}) = 1079.4$ Euros one year after.

(f) With $r_X$ and $r_Y$ being the bond yields of X and Y,

$$D_X = \frac{1\times 13 \times \frac{1}{1+r_X} + 2\times 113 \times \left(\frac{1}{1+r_X}\right)^2}{13 \times \frac{1}{1+r_X} + 113 \times \left(\frac{1}{1+r_X}\right)^2} = \frac{13(1+r_X)+226}{13(1+r_X)+113}$$

$$D_Y = \frac{1\times 5 \times \frac{1}{1+r_Y} + 2\times 105 \times \left(\frac{1}{1+r_Y}\right)^2}{5 \times \frac{1}{1+r_Y} + 105 \times \left(\frac{1}{1+r_Y}\right)^2} = \frac{5(1+r_Y)+210}{5(1+r_Y)+105}.$$

2. Let $\alpha$ be the portion of the asset A in the portfolio. Since short selling is not allowed, $0 \le \alpha \le 1$. Now

$$\bar{r}_p = \alpha \bar{r}_A + (1-\alpha)\bar{r}_B$$

$$\sigma_p^2 = \alpha^2 \sigma_A^2 + (1-\alpha)^2 \sigma_B^2 + 2\alpha(1-\alpha)\rho_{AB}\sigma_A\sigma_B$$

$$\frac{\partial}{\partial \alpha}\sigma_p^2 = 2\alpha\sigma_A^2 - 2(1-\alpha)\sigma_B^2 + 2(1-2\alpha)\rho_{AB}\sigma_A\sigma_B$$

$$= (2\sigma_A^2 + 2\sigma_B^2 - 4\rho_{AB}\sigma_A\sigma_B)\alpha + 2\rho_{AB}\sigma_A\sigma_B - 2\sigma_B^2.$$

(a) If $\rho_{AB} = 1$,

$$\frac{\partial}{\partial \alpha}\sigma_p^2 = 2(\sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B)\alpha + 2\sigma_A\sigma_B - 2\sigma_B^2$$

$$= 2(0.2 - 0.3)^2\alpha + 2\times 0.2\times 0.3 - 2\times 0.3^2$$

$$= 0.02\alpha - 0.06$$

When $0 \le \alpha \le 1$, $\frac{\partial}{\partial\alpha}\sigma_p^2 < 0$ implying that $\sigma_p^2$ decreases monotonically. So we have a minimum when $\alpha = 1$, meaning that we should invest 100% in asset A and none in asset B.

(b) If $\rho_{AB} = 0$,

$$\frac{\partial}{\partial \alpha}\sigma_p^2 = 2(\sigma_A^2+\sigma_B^2)\alpha - 2\sigma_B^2$$

$$= 0.26\alpha - 0.18 = 0$$

$$\Rightarrow \alpha = 0.6923 .$$

$$\frac{\partial^2}{\partial \alpha^2}\sigma_p^2 = 2(\sigma_A^2+\sigma_B^2) > 0 .$$

Thus $\alpha = 0.6923$ which lies between 0 and 1 is the true minimum. This means that we should invest 69.23% in asset A and 30.77 in asset B.

(c) If $\rho_{AB} = -1$,

$$\frac{\partial}{\partial \alpha}\sigma_p^2 = 2(\sigma_A^2+\sigma_B^2+2\sigma_A\sigma_B)\alpha - 2\sigma_A\sigma_B - 2\sigma_B^2$$

$$= 2(0.2+0.3)^2\alpha - 2\times 0.2\times 0.3 - 2\times 0.3^2$$

$$= 0.5\alpha - 0.3 = 0$$

$$\Rightarrow \alpha = 0.6 .$$

$$\frac{\partial^2}{\partial \alpha^2}\sigma_p^2 = 0.5 > 0$$

Thus $\alpha = 0.6$, lying between 0 and 1, is the true minimum. We have thus to invest 60% in asset A and 40% in asset B.

3. (a) Gold in this context acts like a security with a constant dividend yield of 2%. With continuous compounding, the six month forward price is, therefore,

$$300 \exp\left((0.05-0.02)/2\right) = 304.534.$$

(b) Suppose that the lender will receive a cash payment $P$ upon delivery. The PV of an ounce of gold delivered in six month is $300\exp(-0.02/2)$. Setting the PV of the overall cash flow equal to zero, we have

$$0 = -30{,}000 + P\exp\left(-0.05/2\right) + 100 \times 300\exp\left(-0.02/2\right)$$

or, $$P = 30{,}000\exp\left(0.05/2\right) - 30{,}000\exp\left((0.05-0.02)/2\right)$$

$$= €306.06.$$

By using transaction table, as an alternative,

| Trade | Payment today | in six months |
|---|---|---|
| make the loan | −€30,000 | + 100 oz gold |
| sell futures for 100 oz of gold | | 100 × 304.534, −100 oz gold |
| receive additional cash payment | | P |
| total | −€30,000 | €30,453.40 + P |

Discounting at the risk-free rate gives:

$$30{,}000 = (30{,}543.40 + P)e^{-0.05/2}$$

$$\Rightarrow P = 30{,}000\,e^{0.05/2} - 30{,}453.40 = €306.06.$$

4. (a) The parameters are

$$u = e^{0.20\sqrt{1/52}} = 1.0281$$

$$d = e^{-0.20\sqrt{1/52}} = 0.9726$$

The risk-neutral probability of the market going up is

$$q_u = \frac{R-d}{u-d} = 0.5104.$$

(b) The option price is

$$V = \frac{1}{1+r}\left(150 \times q_u + 50(1-q_u)\right)$$

$$= 100.94 .$$

(c) The replicating portfolio consists of $\Delta$ shares of the market, where

$$\Delta = \frac{V_u - V_d}{S(u-d)} = \frac{150-50}{100(u-d)} = 18.0254$$

and an amount $B$ of bonds, where

$$B = V - 100\Delta = -1701.60 .$$

The beta of the option is the same as the beta of the replicating portfolio, which, by the additivity of beta, is the value proportion of the portfolio invested in the market (recall that the beta of the market is one and the beta of the risk-free bond is zero):

$$\beta = \frac{100\Delta}{V} = 17.8569 .$$