Course : **Model Reduction**

Date : April 14, 2026
Time : 13:45-15:45

Motivate all answers and calculations.
The use of electronic devices is not permitted.

[1p] **1.** Given a stable state-space system with controllability and observability Gramians

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad Q = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

Argue whether or not you would reduce this model to a reduced order model with state-space size 1. Justify your answer.

[3p] **2.** Consider the state-space system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t),$$

with

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Derive a reduced order model of state-space size 2 by balanced truncation. How accurate is this reduced order model compared to the full order model? Write down all steps explicitly.

[1p] **3.** Consider the state space equation

$$\dot{x}(t) = Ax(t).$$

We assume that $A = J \in \mathbb{R}^{n \times n}$, with $J^T = -J$. Show that all eigenvalues of A lie on the imaginary axis.

4. Consider the advection equation

$$u_t + vu_x = 0 \;\; ; \;\; 0 < x < L \; , \; t > 0 \tag{1}$$

where $v > 0$ is constant. The initial condition is chosen as $u(x, 0) = f(x)$ and homogeneous Neumann boundary conditions are applied: $u_x(0, t) = u_x(L, t) = 0$. Define the kinetic energy

$$E(t) = \int_0^L \frac{1}{2} u^2(\xi, t) \, d\xi \tag{2}$$

[1p] a) Show that the kinetic energy is conserved.

The spatial derivative is approximated on a uniform grid with grid-spacing $h = L/N$ where N denotes the number of grid cells. For this discretization we adopt

$$u_x(x_j, t) \approx \delta_x U_j(t) := \frac{1}{h}(U_{j+1}(t) - U_j(t)) \tag{3}$$

where $U_j(t)$ denotes the approximate solution at x_j as a function of time.

[1p] b) Derive the truncation error for this discretisation and determine the order of accuracy.

[1p] c) What is the modified equation corresponding to this discretization?

[1p] d) Determine the evolution of E corresponding to this modified equation. Interpret your result.

Total: 1+2.5+1.5+4 = 9 points
Grade 1+ number of points