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University of Twente
Q1 2025

Analysis 3: Resit 08-11-2025

The basic time allocated for this exam is 3 hours (8:45-11:45)
No books, cheatsheets or electronic devices are allowed.
All answers must be justified, unless a question is explicitly marked as "final answer"

Question 1 (4p). Find the boundary of the set

$$\mathbb{Q} \times (0,1) \subseteq \mathbb{R}^2,$$

in the metric space $(\mathbb{R}^2, d_2)$, where d_2 is the usual Euclidean distance.

Question 2 (8p). Find the closure of the set of real sequences with finitely many nonzero elements

$$c_{00} = \big\{(x_n)_{n\in\mathbb{N}} \, : \, \exists N \text{ such that } x_n = 0 \; \forall n \geq N\big\},$$

in the metric space of bounded real sequences

$$\ell^\infty = \big\{(x_n)_{n\in\mathbb{N}} \, : \, \sup_n |x_n| < +\infty\big\},$$

equipped with the distance $d_\infty\big((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\big) = \sup_{n\in\mathbb{N}} |x_n - y_n|$.

Question 3 (8p). Is the following statement true? Give a proof or a counterexample:

Let (M, d_M) and (N, d_N) be metric spaces, $f : M \to N$ a uniformly continuous function, and $B \subseteq N$ connected. Then $f^{-1}(B)$ is also connected.

Remember: We defined connectedness for metric spaces, so B is considered with the metric restricted from N, and $f^{-1}(B)$ with the metric restricted from M.

Question 4 (5+6p). Consider the normed space $(\mathrm{C}[0,1], \|\cdot\|_\infty)$ of continuous real-valued functions on $[0,1]$, with $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$ for all $f \in \mathrm{C}[0,1]$.

(a) Show that for any $r > 0$, the closed unit ball $K_r(0)$ in this space is not compact.

(b) Show that there is a unique function $f \in \mathrm{C}[0,1]$ such that

$$f(x) = x + \int_0^1 \frac{|x-y|}{2} f(y) \, \mathrm{d}y \quad \forall x \in [0,1].$$

Question 5 (6+6p). Consider the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathbf{f}(x_1, x_2) := \left(x_2\, \mathrm{e}^{\cos(x_1+x_2)},\ \sin(x_1)\right)^\top.$$

(a) Use the inverse function theorem to show that there is a local inverse to $\mathbf{f}$ which is defined on an open set $W \subseteq \mathbb{R}^2$ such that $\mathbf{f}(\pi/4, \pi/4) \in W$.

(b) Show that there is some $r > 0$ such that $\mathbf{f}$ has Lipschitz constant at most 2 on the open ball of radius r with center $(\pi/4, \pi/4)^\top$, that is

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\|_2 \leq 2\|\mathbf{x} - \mathbf{y}\|_2 \quad \forall \mathbf{x}, \mathbf{y} \in B_r^{d_2}\left((\pi/4, \pi/4)^\top\right).$$

Hint: The inequality $\|\mathbf{z}\|_2 \leq \sqrt{p}\,\|\mathbf{z}\|_\infty$ for vectors $\mathbf{z} \in \mathbb{R}^p$ can help.

Question 6 (10p). Let $E \subseteq \mathbb{R}^2$ be a Jordan region, and $a < b$ be real numbers. Show, using directly the definitions in terms of volume zero and grids, that the set $E \times [a, b] \subseteq \mathbb{R}^3$ is also a Jordan region.

Question 7 (3+6p). (a) State the Gauss theorem.

(b) Use it to find the oriented surface integral

$$\int_{\psi(E)} \mathbf{F} \cdot \mathbf{N}_\psi \, \mathrm{d}A$$

of the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\mathbf{F}(x_1, x_2, x_3) = \left(\mathrm{e}^{x_2} x_3,\ \mathrm{e}^{x_1} x_3,\ 1 - x_3\right)^\top,$$

on the cylinder parametrized by $\psi : E \to \mathbb{R}^3$ with

$$E = [0, 2\pi] \times [0, 1], \quad \text{and} \quad \psi(u_1, u_2) = \left(\cos(u_1), \sin(u_1), u_2 - 1\right)^\top.$$

Question 8 (8p). Consider the piecewise C^1 curve C in $\mathbb{R}^3$ given by line segments between the points $(1,0,0)^\top, (0,1,0)^\top, (-1,0,0)^\top$ and $(0,-1,0)^\top$, with parametrizations such that the vertices are traversed in that order. Find the oriented line integral

$$\int_C \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s,$$

for the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\mathbf{F}(x_1, x_2, x_3) = \left(x_2 \cos(x_1 x_2) - x_2,\ x_1 \cos(x_1 x_2),\ x_3\right)^\top.$$

Grading.
Best of the following:

- (70 Points in this resit)/10 + (sum of 3 best presentations/hand-ins)
- (70 Points in this resit)/7
- The final grade you had after the exam (with or without presentations/hand-ins)
