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University of Twente
Q1 2025

Analysis 3: Exam 24-10-2025

The basic time allocated for this exam is 3 hours (8:45-11:45)
No books, cheatsheets or electronic devices are allowed.
All answers must be justified, unless a question is explicitly marked as "final answer"

Question 1 (3p+6p). Find:

(a) The boundary ∂A of

$$A = \Big(\big([-1,0) \times [-1,1]\big) \cup \big((0,1] \times [-1,1]\big)\Big) \cap \mathbb{Q}^2$$

in the metric space $(\mathbb{Q}^2, d_0)$, where d_0 is the discrete metric.

(b) The closure of the set of bounded and strictly increasing sequences

$$I = \big\{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \ : \ x_n < x_m \text{ whenever } n < m\big\}$$

in the metric space ℓ^∞ of bounded real sequences with the distance induced by the norm $\|(x_n)_{n\in\mathbb{N}}\|_\infty := \sup_{n\in\mathbb{N}} |x_n|$.

Question 2 (8p). Consider the set and metric

$$M = \big(\{0\} \times [-1,1]\big) \cup \big([-1,1] \times \{0\}\big) \subseteq \mathbb{R}^2, \quad \text{and} \quad d_2(\mathbf{x},\mathbf{y}) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}.$$

Is the metric space (M, d_2) connected?

Question 3 (8p). Consider the set $M = [0, +\infty) \cup \{+\infty\}$ and the two functions $f : M \to [0,1]$ and $d : M \times M \to [0, +\infty)$ given by

$$f(x) = \begin{cases} \frac{x}{1+x} & \text{if } x \in [0,+\infty) \\ 1 & \text{if } x = +\infty \end{cases}, \quad d(x,y) = \big|f(x) - f(y)\big|.$$

Is the metric space (M, d) compact?
Note: it is assumed that (M,d) is a metric space, so no need to check that.

Question 4 (8p). Let H be a Hilbert space with inner product and norm denoted by $\langle \mathbf{x}, \mathbf{y}\rangle$ and $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle}$ for $\mathbf{x}, \mathbf{y} \in H$, and let $\varphi : H \to \mathbb{R}$ be linear and continuous with $\|\varphi\| = 1$. Show that there is a unique $\mathbf{y} \in \{\mathbf{x} \in H : \|\mathbf{x}\| = 1\}$ with $\varphi(\mathbf{y}) = 1$.

Question 5 (10p). Let $\mathbf{f} : \mathbb{R}^4 \to \mathbb{R}^2$ be the function defined as

$$\mathbf{f}(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_2 \sin(x_3) \\ x_1 x_2 - x_3 x_4 \end{pmatrix}.$$

- Show that there is an open set $W \subseteq \mathbb{R}^2$ with $(1, -1)^\top \in W$ and a function $\mathbf{g} : W \to \mathbb{R}^2$ such that $\mathbf{g}(1, -1) = (1, 0)^\top$ and

$$\mathbf{f}\big(y_1, g_1(y_1, y_2), g_2(y_1, y_2), y_2\big) = (0, 1)^\top \text{ for all } \mathbf{y} = (y_1, y_2)^\top \in W,$$

where g_1, g_2 are the components of $\mathbf{g}$, that is $\mathbf{g}(\mathbf{y}) = (g_1(\mathbf{y}), g_2(\mathbf{y}))^\top$.
- Find $D\mathbf{g}(1, -1)$.

Question 6 (10p). Show that the function $f : [0, 1]^2 \to \mathbb{R}$ defined by

$$f(x_1, x_2) = \begin{cases} x_2 & \text{if } x_2 > \sqrt{1 - x_1^2} \\ 0 & \text{if } x_2 \leq \sqrt{1 - x_1^2} \end{cases}$$

is Riemann integrable.

Question 7 (12p). Consider the rectangles $E_1 = [0, 2\pi] \times [0, 1]$, $E_2 = [0, 2\pi] \times [-1, 0]$, $\psi_j : E_j \to \mathbb{R}^3$ for $j = 1, 2$ given by

$$\psi_j(u_1, u_2) = \left(\left(1 + (-1)^j \frac{u_2}{2}\right) \cos u_1,\ \left(1 + (-1)^j \frac{u_2}{2}\right) \sin u_1,\ u_2\right)^\top,$$

and the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\mathbf{F}(x_1, x_2, x_3) = \left(1 - x_3^2,\ 1 - x_3^2,\ x_3\right)^\top.$$

Find the oriented surface integral

$$\int_S \mathbf{F} \cdot \mathbf{N} \, \mathrm{d}A,$$

with $S = \psi_1(E_1) \cup \psi_2(E_2)$ and normal vector $\mathbf{N}$ induced by these parametrizations.

Question 8 (5p). Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by

$$\mathbf{F}(x_1, x_2, x_3) = (-x_1,\ x_2 x_3,\ x_3)^\top.$$

Justify if there can be a C^2 vector field $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathbf{F} = \operatorname{curl} \mathbf{G}$ and if so, find one.

Grading.

Best of the following:

- (70 Points in this exam)/10 + (sum of 3 best presentations/hand-ins)
- (70 Points in this exam)/7