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University of Twente
Q1 2024

Analysis 3: Resit 06-11-2024

The basic time allocated for this exam is 3 hours (8:45-11:45)
No books, cheatsheets or electronic devices are allowed.
All answers must be justified, unless a question is explicitly marked as "final answer"

Question 1 (3p). Define $d : [0,2] \times [0,2] \to [0,+\infty)$ by

$$d(x,y) = \min\left\{1, |x-y|^2\right\}.$$

Is $([0,2], d)$ a metric space?

Question 2 (2p+2p). Justifying your answers, find

(a) The boundary ∂A of

$$A = \left\{(x_1, x_2)^\top : x_1 \notin \mathbb{Z},\ x_2 \in \mathbb{Z}\right\} \quad \text{in } (\mathbb{R}^2, d_2).$$

(b) The cluster points of

$$B = \left\{f \in C[0,1] \,:\, f(1/2) \in \mathbb{Q}\right\} \quad \text{in } (C[0,1], d_\infty),$$

where $C[0,1]$ is the set of continuous functions from $[0,1]$ to $\mathbb{R}$, with the metric $d_\infty(f,g) := \max_{t\in[0,1]} |f(t) - g(t)|$.

Question 3 (3p). Let (M, d_M) and (N, d_N) be two metric spaces, and $f : M \to N$ be uniformly continuous.

Prove that if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in M, then $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in N.

Question 4 (4p). Let (M, d) be a metric space and $A, B \subseteq M$ be compact.

Prove that $A \cup B$ is also compact.

Question 5 (5p). Let $(X, \|\cdot\|_X)$ be a normed vector space, and a function $f : X \to \mathbb{R}$ be differentiable at the zero vector $\mathbf{0} \in X$, with $f(\mathbf{0}) = 0$, but for which the bounded linear map $Df(\mathbf{0}) : X \to \mathbb{R}$ is not identically zero.

Prove that for every $\delta > 0$ there is $\mathbf{x}_\delta \in B_\delta(\mathbf{0}) \subseteq X$ for which $f(\mathbf{x}_\delta) > 0$.

Question 6 (4p). Let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$ be the function defined as

$$\mathbf{f}(x_1, x_2, x_3) = \begin{pmatrix} x_1x_2 - x_3 \\ x_3x_1^2 - x_2^2x_1^2 \end{pmatrix}.$$

Show that there is an open set $W \subseteq \mathbb{R}$ such that $2 \in W$ and a function $\mathbf{g} : W \to \mathbb{R}^2$ with $\mathbf{g}(2) = (2, 4)^\top$ and

$$\mathbf{f}\big(g_1(y), y, g_2(y)\big) = 0 \text{ for all } y \in W,$$

where g_1, g_2 are the components of $\mathbf{g}$, that is $\mathbf{g}(y) = (g_1(y), g_2(y))^\top$.
Moreover, find $D\mathbf{g}(2)$.

Question 7 (4p). Let $E \subseteq \mathbb{R}^2$ be given by

$$E = \Big\{ (x_1, x_2)^\top : x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 1 \Big\}.$$

Show, directly from the definition in terms of grids, that E is a Jordan region.

Question 8 (4p). Let $\gamma : [0, 3] \to \mathbb{R}^2$ be given by

$$\gamma(t) = \begin{cases} (t, t^2 - 1)^\top & \text{for } t \in [0, 1), \\ (2 - t, 0)^\top & \text{for } t \in [1, 2], \\ (0, 2 - t)^\top & \text{for } t \in [2, 3], \end{cases}$$

and $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\mathbf{F}(x_1, x_2) = \left(\frac{1}{1 + x_1^2}, x_1^2 \right)^\top.$$

Denoting $C = \gamma([0, 3])$, use the Green theorem to compute the oriented line integral

$$\int_C \mathbf{F} \cdot \mathbf{T}_\gamma \, ds.$$

Question 9 (2p+2p). Let $\mathbf{F}, \mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector fields given by

$$\mathbf{F}(x_1, x_2, x_3) = (x_1^2 + x_3^2, 0, 2x_1x_3)^\top \text{ and } \mathbf{G}(x_1, x_2, x_3) = (3x_1x_3, -3x_2^2, -x_1)^\top.$$

Justify if there can be a C^2 scalar function $f : \mathbb{R}^3 \to \mathbb{R}^{\overset{1}{3}}$ such that $\mathbf{F} = \nabla f$ and if so, find one. Do the same for C^2 functions $g : \mathbb{R}^3 \to \mathbb{R}^{3 1}$ such that $\mathbf{G} = \nabla g$.

Grading.
Best of the following:

- (35 Points in this resit)/5 + (Sum of best 5 of tutor session+hand-in)*3/50
- (35 Points in this resit)/3.5
- Your previous grade after the first exam (incl. tutor sessions+hand-in).