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University of Twente
Q1 2024

Analysis 3: Exam 23-10-2024

The basic time allocated for this exam is 3 hours (8:45-11:45)
No books, cheatsheets or electronic devices are allowed.
All answers must be justified, unless a question is explicitly marked as "final answer"

Question 1 (2p). Let (M_1, d_1) and (M_2, d_2) be two metric spaces, and define a function $D : (M_1 \times M_2) \times (M_1 \times M_2) \to [0, +\infty)$ by

$$D(\mathbf{x}, \mathbf{y}) = \max\left\{d_1(x_1, y_1),\, d_2(x_2, y_2)\right\} \text{ where } \mathbf{x} = (x_1, x_2),\ \mathbf{y} = (y_1, y_2).$$

Show that $(M_1 \times M_2, D)$ is also a metric space.

Question 2 (2p+2p, final answer). Find the closure and interior of the following subsets of metric spaces:

$$A = \left\{(x_1, x_2)^\top \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 1 \text{ and } x_2 \neq 0\right\} \quad \text{in } (\mathbb{R}^2, d_2),$$
$$B = \mathbb{Q} \cap (0, 1) \quad \text{in } (\mathbb{R}, d_0),$$

where d_0 is the discrete metric.

Question 3 (4p). Let $S \subseteq \mathbb{R}$ be given by with $d_2(x,y) = |x-y|$

$$S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}.$$

Show that S is compact.

Question 4 (5p). Let (M, d_M) and (N, d_N) be two metric spaces, and $f : M \to N$ be continuous, surjective, and such that

$$d_M(x, y) \le d_N\big(f(x), f(y)\big) \quad \text{for all } x, y \in M.$$

Show that if (M, d_M) is complete, then (N, d_N) is also complete.

Question 5 (3p+2p+3p). Consider the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathbf{f}(x_1, x_2) := \begin{cases} \left(x_2^2 \overset{\ln}{\log}(x_1^2 + x_2^2),\ x_1\right)^\top & \text{if } (x_1, x_2)^\top \neq (0,0)^\top \\ (0,0)^\top & \text{if } (x_1, x_2)^\top = (0,0)^\top. \end{cases}$$

(a) Is $\mathbf{f}$ differentiable at all points $\mathbf{x} \in \mathbb{R}^2$?

(b) State the inverse function theorem for a function $\mathbf{g} : \mathcal{O} \to \mathbb{R}^2$ with $\mathcal{O} \subseteq \mathbb{R}^2$ open.

(c) Use the inverse function theorem to find a local inverse to $\mathbf{f}$ which is defined on an open set containing the point

$$\mathbf{y}_0 = \left(\frac{\mathrm{e}}{2}, \sqrt{\frac{\mathrm{e}}{2}}\right),$$

and compute the Jacobian matrix of this local inverse at $\mathbf{y}_0$.

Question 6 (4p). Let $E \subseteq \mathbb{R}^2$ be given by

$$E = \left\{ \left(\frac{1}{n}, \frac{1}{2n}\right)^\top : n \in \mathbb{N} \right\} \cup \left\{ (0,0)^\top \right\}.$$

Show that E is of volume zero.

Question 7 (4p). Let $\gamma : [0, \pi] \to \mathbb{R}^3$ and $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$\gamma(t) = \left(\sin(t), \sin(t), \cos(t)\right)^\top,$$
$$\mathbf{F}(x_1, x_2, x_3) = \left(3x_1^2 + x_2^3 + x_3^3,\ x_1^3 + 3x_2^2 + x_3^3,\ x_1^3 + x_2^3 + 3x_3^2\right)^\top.$$

Denoting $C = \gamma([0, \pi])$, evaluate the oriented line integral

$$\int_C \mathbf{F} \cdot \mathbf{T}_\gamma \, \mathrm{d}s.$$

Question 8 (4p). Let $E \subseteq \mathbb{R}^3$ be defined by

$$E = \left\{ (x_1, x_2, x_3)^\top : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1, x_3 \in [0, 1] \right\},$$

and denote by S its topological boundary ∂E oriented negatively (that is, with unit normal vector $\mathbf{N}$ pointing towards E). For $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\mathbf{F}(x_1, x_2, x_3) = \left(\sqrt{x_2},\ x_2 x_3^2,\ x_1^2\right)^\top,$$

evaluate the oriented surface integral

$$\int_S \mathbf{F} \cdot \mathbf{N} \, \mathrm{d}A.$$

Grading: [35 Points in this exam]/5 + [Sum of best 5 of tutor session+hand-in]*3/50